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Galois Actions on ℓ -adic Local Systems and Their Nearby Cycles: A Geometrization
of Fourier Eigendistributions on the p -adic Lie Algebra $\mathfrak{sl}(2)$

by

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A THESIS

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Abstract

In this thesis, two $\overline{\mathbb{Q}_\ell}$ -local systems, ${}^\circ\mathcal{E}$ and ${}^\circ\mathcal{E}'$ (Definition 3.2.1) on the regular unipotent subvariety $\mathcal{U}_{0,K}$ of p -adic $\mathrm{SL}(2)_K$ are constructed. Making use of the equivalence between $\overline{\mathbb{Q}_\ell}$ -local systems and ℓ -adic representations of the étale fundamental group, we prove that these local systems are equivariant with respect to conjugation by $\mathrm{SL}(2)_K$ (Proposition 3.3.5) and that their nearby cycles, when taken with respect to appropriate integral models, descend to local systems on the regular unipotent subvariety of $\mathrm{SL}(2)_k$, k the residue field of K (Theorem 4.3.1). Distributions on $\mathrm{SL}(2, K)$ are then associated to ${}^\circ\mathcal{E}$ and ${}^\circ\mathcal{E}'$ (Definition 5.1.4) and we prove properties of these distributions. Specifically, they are admissible distributions in the sense of Harish-Chandra (Proposition 5.2.1) and, after being transferred to the Lie algebra, are linearly independent eigendistributions of the Fourier transform (Proposition 5.3.2). Together, this gives a geometrization of important admissible invariant distributions on a nonabelian p -adic group in the context of the Local Langlands program.

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Introduction

Here is what happens in this thesis in a rough nutshell: with a pair of important admissible invariant distributions on p -adic $\mathrm{SL}(2)$ in mind, a pair of equivariant $\overline{\mathbb{Q}_\ell}$ -local systems on the étale site of p -adic $\mathrm{SL}(2)$ are produced. Then it is shown how these sheaves encode all of the information held in the distributions. In sum, this provides a new kind of geometrization in the context of the local Langlands program. There are several crossing threads in that description; let us start pulling them.

The first is the study of representations of p -adic groups, which can be said to have begun with Mautner’s “*Spherical functions over p -adic fields*” [28] in 1958, and has enjoyed a remarkably fertile half century since then. Ultimately, the representations of a p -adic group G we want to determine are the smooth representations, which are those $\pi: G \longrightarrow \mathrm{Aut}(V)$ such that $V = \cup V^K$, where K ranges over all open compact subgroups of G . If each V^K is also finite dimensional, the representation is called admissible. In that paper and its sequel [29], Mautner introduced two important classes of smooth representations, including supercuspidal representations. All irreducible smooth representations of a p -adic group G are subquotients of representations induced from supercuspidal representations of parabolic subgroups of G .

Distributions enter this picture with the work of Harish-Chandra. Among other things, he developed and introduced the appropriate notion of a character of an admissible representation as a distribution on the Hecke algebra of locally constant and compactly supported functions on G [14]. The intimate connection this established between the representation theory of p -adic groups and harmonic analysis only deepened in the wake of Langlands's work, particularly the famous conjectural correspondence that bears his name.

A rough (and admittedly nonstandard) way of describing the Local Langlands Correspondence is to say that it promises a function between characters of irreducible admissible representations to certain representations (called L-parameters) of the Galois group of the local field in play. In general this function is not injective, but it is predicted that its fibres are parametrized by a specific finite group. The finite set of representations whose characters appear together in the fibre above a single parameter is called an L-packet of representations.

The particular distributions at hand in this thesis appear in the vector space spanned by the characters of the admissible representations in the unique L-packet of size 4 for p -adic $\mathrm{SL}(2)$ and possess other remarkable properties. They are stably conjugate; they are supported on topologically unipotent elements; and, when transported to the Lie algebra, they are linearly independent eigendistributions—with the same eigenvalue—for the Fourier transform (in fact, they span the unique eigenspace for the Fourier transform on the Lie algebra of p -adic $\mathrm{SL}(2)$, but that is not proved here). What is achieved in this thesis is a geometrization of these distributions: they are turned into sheaves. Which is the other major thread running through the background of this thesis.

An example may make this notion of ‘geometrization’ more concrete. Let G be an algebraic group, commutative and connected, over a finite field \mathbb{F}_q . The Lang map, $x \mapsto \frac{\text{Frob}(x)}{x}$, defines an étale cover of G with automorphism group isomorphic $G(\mathbb{F}_q)$. Therefore, the étale fundamental group $\pi_1(G, \bar{g})$ (described in Section 1.1 of Chapter 1) comes equipped with a map $\pi_1(G, \bar{g}) \rightarrow G(\mathbb{F}_q)$. Combined with a character $\theta: G(\mathbb{F}_q) \rightarrow \overline{\mathbb{Q}}_\ell^\times$ and the canonical map $\pi_1(\bar{G}, \bar{g}) \rightarrow \pi_1(G, \bar{g})$, where $\bar{G} = G \times_{\text{Spec}(\mathbb{F}_q)} \text{Spec}(\overline{\mathbb{F}}_q)$, we get a character of $\pi_1(\bar{G}, \bar{g})$:

$$\pi_1(\bar{G}, \bar{g}) \rightarrow \pi_1(G, \bar{g}) \rightarrow G(\mathbb{F}_q) \xrightarrow{\theta} \overline{\mathbb{Q}}_\ell^\times.$$

This character is equivalent to an ℓ -adic local system \mathcal{L} on \bar{G} possessing an isomorphism

$$\phi: \text{Frob}^* \mathcal{L} \xrightarrow{\sim} \mathcal{L}.$$

For any \mathbb{F}_q -point \bar{g} of G , the composition of the canonical isomorphism

$$\mathcal{L}_{\bar{g}} = \mathcal{L}_{\text{Frob}(\bar{g})} \cong (\text{Frob}^* \mathcal{L})_{\bar{g}}$$

with the map on stalks induced by ϕ gives an automorphism of $\mathcal{L}_{\bar{g}}$ whose trace belongs to $\overline{\mathbb{Q}}_\ell^\times$. The function $t_{\text{Frob}}^{\mathcal{L}}: G(\mathbb{F}_q) \rightarrow \overline{\mathbb{Q}}_\ell^\times$ this defines is equal to θ .

The local system \mathcal{L} is in fact an example of a character sheaf, which is the foremost example of geometrization in representation theory. George Lusztig introduced character sheaves in a series of papers that began to appear in 1985 [21, 22, 23, 24, 25]. In their original iteration, character sheaves were defined on general linear groups over an algebraically closed field of positive characteristic. From them, Lusztig was

able to obtain, by means of the Grothendieck-Lefschetz trace formula, all irreducible representations of $\mathrm{GL}(n, \mathbb{F}_{p^m})$ for all n and m , uniting the representation theory of all these groups in an object related to the geometry of $\mathrm{GL}(n)$.

Lusztig has gone on to generalize character sheaves to larger and larger classes of groups [20], and despite the fact that they are only ever applied to groups over algebraically closed fields of positive characteristic, their definition makes sense even for groups over fields of characteristic 0. Further expansions of the character sheaf perspective have been carried out by, for example, Drinfeld and his collaborators (in positive characteristic), and by Cunningham and Salmasian, who have taken steps to develop character sheaves for use in characteristic 0 [7].

As one might expect, this latter effort meets extra difficulty. For example, if G is a linear algebraic group defined over the generic fibre of a Henselian trait, accessing Lusztig’s methods—or more to the point, accessing the trace formula—requires moving whatever geometric objects one decides to work with from G to the special fibre of an integral model for G (or some scheme closely related to G). A choice of integral model for G makes this possible. And, fortunately, after making such a choice there is machinery suited to moving sheaves from the generic fibre to the special fibre: Deligne’s adaptation of the nearby cycles functor to schemes defined over Henselian traits. As described in SGA 7(II) [1], the nearby cycles functor is robust enough to even allow the addition of Galois actions to the sheaves one chooses to study, making it possible to work even on groups defined over algebraically unclosed fields. Yet another degree of freedom is offered by the multiplicity of integral models that can be chosen to use in the definition of the functor.

With all this, what remains is finding appropriate geometric objects (by which

we always mean sheaves of some kind or other) on G so that, when fed into this machinery, it reproduces aspects of the representation theory of p -adic groups as it is known to us presently. One use of this perspective is to try to replace the basic objects on both sides of the Local Langlands Correspondence with a geometric avatar, which would allow the techniques of algebraic geometry to be brought to bear on the correspondence itself. Recent work (in preparation) by Achar, Cunningham, Kamgarpour, and Salmasian has found how to geometrize the Galois side of the Local Langlands Correspondence. This thesis establishes a result in this spirit, which is what we mean by ‘geometrization in the context of the local Langlands program.’

Specifically, we work on $G = \mathrm{SL}(2)_K$, where K is a p -adic field with finite residue field k of positive characteristic. In Chapter 3, we define two local systems on the regular unipotent subvariety $\mathcal{U}_{0,K}$, which are our chosen geometric objects on G , and equip their base change to $\mathcal{U}_{0,\overline{K}}$ with an action of $\mathrm{Gal}(\overline{K}/K)$. In fact, these local systems are very special: when extended by zero from $\mathcal{U}_{0,K}$, they define two perverse sheaves on G that are identical after change of base to \overline{K} . That perverse sheaf on \overline{G} is the unique cuspidal unipotent character sheaf for \overline{G} . Deploying the heavy machinery of perverse sheaves and character sheaves is not necessary to prove the results in this thesis, so these facts are not proved here. But the results we do obtain show that the unique cuspidal unipotent character sheaf for \overline{G} can be equipped with two very different Galois actions, and shows how the resulting Galois sheaves determine two very different, and very special, distributions on $G(K)$.

These result are contained in the remainder of Chapter 3 and in Chapter 4, where important properties of the pair of local systems and their nearby cycles are shown. In particular, it is shown that the local systems are equivariant with respect to the

conjugation action of G on itself (Proposition 3.3.5), and that with respect to certain integral models for $\mathcal{U}_{0,K}$ their nearby cycles—which are local systems on $\mathcal{U}_{0,\bar{k}}$ equipped with an action of $\mathrm{Gal}(\bar{K}/K)$ —descend to local systems on $\mathcal{U}_{0,k}$ (Theorem 4.3.1). Finally, in Chapter 5, we apply all those results by defining, with the aid of the Grothendieck-Lefschetz trace formula, distributions on the Hecke algebra of $G(K)$ (and $\mathfrak{g}(K)$, \mathfrak{g} the Lie algebra of G). These distributions are shown to be linearly independent invariant admissible distributions that are eigendistributions of the Fourier transform (Propositions 5.2.1 and 5.3.2). Together, this establishes a successful geometrization.

Before all that, the first two chapters are spent providing background for the work of Chapter 3. Chapter 1 gives the definitions and basic properties of π -adic sheaves and $\overline{\mathbb{Q}}_\ell$ -local systems, which are the basic categories underlying the geometric objects at the centre of the thesis. Despite the name, these are not honest sheaves, but projective systems composed of sheaves. As such, they are somewhat cumbersome, but, as was only hinted at in the example above, there is an equivalence of categories between $\overline{\mathbb{Q}}_\ell$ -local systems and ℓ -adic representations of the étale fundamental group. Accordingly, the étale fundamental group is defined and this equivalence is described in Chapter 1. Chapter 2 recalls Deligne’s study of Galois actions on sheaves in SGA [1, Exposé XIII] and describes its extension to $\overline{\mathbb{Q}}_\ell$ -local systems, which is a crucial ingredient of the results in Chapter 4. The opening sections of Chapter 4 recall the other crucial ingredient, the nearby cycles functor.

Altogether, this thesis represents the first example of geometrization, in the context of the local Langlands program, of sophisticated admissible invariant distributions on nonabelian p -adic groups. As such, it invites a question: can this be done

more generally? Already, joint work with Aubert and Cunningham (in preparation) has found that the techniques employed in this thesis can indeed be used to geometrize other distributions on other p -adic groups. However, those distributions are also eigendistributions of the Fourier transform and, as such, represent a relatively small class. And, the local systems that appear as geometric avatars of distributions in that class are, like our local systems, closely related to cuspidal unipotent character sheaves. So, is it possible to geometrize all admissible invariant distributions on p -adic groups? While the answer is not known, and certainly depends on what is meant by geometrization, the work here points a way forward: by combining the techniques developed in this thesis with recent work by Cunningham and Roe on the geometrization of characters of p -adic tori, it appears that it may be possible to find geometric avatars for a significant class of admissible invariant distributions, and in doing so, shed light on the Local Langlands Correspondence, one of the most important open problems in number theory.

Chapter 1

The Étale Fundamental Group, π -adic Sheaves, $\overline{\mathbb{Q}}_\ell$ -Local Systems

The main results of this thesis concern a pair of $\overline{\mathbb{Q}}_\ell$ -local systems defined on the smooth subvariety of regular unipotent elements in $\mathrm{SL}(2)_K$. The category of $\overline{\mathbb{Q}}_\ell$ -local systems is obtained from the category of π -adic sheaves. These are somewhat complicated and unwieldy objects, so it is our preference to exploit an equivalence between $\overline{\mathbb{Q}}_\ell$ -local systems and the category of ℓ -adic representations of the étale fundamental group, and work instead with the representations rather than the local systems themselves. Therefore, this chapter contains information on the étale fundamental group, π -adic sheaves, $\overline{\mathbb{Q}}_\ell$ -local systems tailored toward understanding the equivalence of categories that will be used throughout the rest of thesis. Because all the results in this chapter are standard, we avoid giving proofs except when such details help with understanding some aspect of the work in later chapters, but do provide references to works where complete proofs can be found.

1.1 Étale Fundamental Group

The theory of the étale fundamental group originated with Grothendieck, and the definitive reference for all the material in this section are the volumes of SGA (in

particular SGA 1 [13]). Another reference, leaner and better suited to the needs of this thesis, is Szamuely's *Galois Groups and Fundamental Groups* [33].

1.1.1 Étale Covers

A morphism of arbitrary schemes $f: Y \rightarrow X$ is *étale* if it is of finite type, flat, and unramified¹. An *étale cover* is an étale morphism that is also finite and surjective. The category of étale covers of a scheme X will be denoted $\text{Fet}(X)$.

Some basic properties enjoyed by étale morphisms and étale covers, all but immediate from the definition (the reader may also consult Milne [30, Ch. 1]), are:

- The composite of two étale morphisms (resp. covers) is also an étale morphism (resp. cover).
- If $g \circ f$ and g are étale morphisms (resp. covers), then f is also an étale morphism (resp. cover).
- If $f: Y \rightarrow X$ is étale, and $g: Z \rightarrow X$ is any morphism, the map $f^Z: Y \times_X Z \rightarrow Z$ obtained by pullback (the *base change* of f by g) is again étale. The same is true for étale covers.

Example 1.1.1. 1. Some examples are immediately obvious: any scheme is an étale cover of itself, as is the disjoint union of finitely many copies of that scheme. Étale covers of this form are called *trivial*. A scheme with no nontrivial étale covers is *simply connected*.

2. Étale covers of the spectrum of a field K are all by spectra of finite products of finite separable extensions of K (which are quite sensibly called finite étale K -algebras).

¹Along with a basic familiarity with algebraic geometry, we elect to assume the reader's familiarity with these terms. In any event, they are fully explained in the sources listed above.

3. The connected étale covers of the multiplicative group scheme

$$\mathbb{G}_{m,K} = \operatorname{Spec} (K[x, y]/(xy - 1))$$

when K is algebraically closed are, after passing to coordinate rings, of the form

$$\begin{aligned} [d]: K[x, y]/(xy - 1) &\longrightarrow K[x, y]/(xy - 1) \\ x &\longmapsto x^d \\ y &\longmapsto y^{-d} \end{aligned}$$

for any $d \in \mathbb{Z}$.

When K is not algebraically closed, the étale covers are

$$\begin{aligned} [d]_F: K[x, y]/(xy - 1) &\longrightarrow F[x, y]/(xy - 1) \\ K &\hookrightarrow F \\ x &\longmapsto x^d \\ y &\longmapsto y^{-d} \end{aligned}$$

for any integer d and finite separable extension $K \hookrightarrow F$.

4. More examples emerge in the wake of the following fact: for an integral normal scheme X with function field F , every connected étale cover of X arises as the normalization of X in a finite separable extension of F .² But not every finite separable extension of K yields an étale cover; the resulting morphism may be ramified. This fact combined with Minkowski's theorem shows that there

²See Lenstra [19, Thm. 6.13] for a proof.

are no nontrivial étale covers of $\mathrm{Spec}(\mathbb{Z})$, nor are there any of \mathbb{A}_K^1 when K is algebraically closed, which accords with the simple connectedness of the affine line over \mathbb{C} .

Beyond the properties listed, étale covers possess, in parallel with ordinary topological covering spaces, a local triviality property. Namely, if $f: Y \rightarrow X$ is an étale cover and X is connected, then there exists an étale cover $g: Z \rightarrow X$ such that $Y \times_X Z$ is isomorphic to a disjoint union of finitely many copies of Z .³ This fact is the foundation that the categorical equivalence between local systems and fundamental group representations is built upon.

1.1.2 Finite Locally Constant Sheaves

A sheaf \mathcal{F} on the étale site of a scheme X is *finite locally constant* if there exists a covering by étale opens⁴ $\{X_i \rightarrow X \mid i \in I\}$ such that $\mathcal{F}|_{X_i}$ is constant with finite stalks. This is equivalent to the set of sections of $\mathcal{F}|_{X_i}$ over any étale open being finite. We will write $\mathrm{Flc}(X)$ for the category of finite locally constant sheaves on X .

There is functor from $\mathrm{Fet}(X)$ to $\mathrm{Flc}(X)$, given by associating to every étale cover $f: Y \rightarrow X$ its sheaf of sections, \mathcal{F}_Y . For each étale scheme U over X ,

$$\mathcal{F}_Y(U) := \{\sigma: U \rightarrow Y \mid f^U \circ \sigma = \mathrm{id}_U\}.$$

If we restrict attention to each connected component of X , that \mathcal{F}_Y is a sheaf of *finite* sets is just a consequence of the fact that f is a finite morphism; local constancy is a result of the local triviality of f together with the fact that any section of an

³This is proved in Szamuely [33, Prop. 5.2.9].

⁴Meaning that, taken together, the images of the maps $X_i \rightarrow X$ are surjective onto the underlying topological space of X .

étale cover induces an isomorphism of connected components of the base space onto connected components of the cover⁵. Descent techniques whose description would lead too far afield provide a quasi-inverse to this functor, proving an equivalence of categories that we record here for reference later⁶:

Proposition 1.1.2. *The category $\text{Fet}(X)$ of a scheme X is equivalent to the category $\text{Flc}(X)$ of finite locally constant sheaves of sets on X .*

With this equivalence in hand, the connection between finite locally constant sheaves and the étale fundamental group is a relatively simple matter to establish because the étale fundamental group classifies étale covers, just as the topological fundamental group classifies topological covers. Therefore, we turn to describing the étale fundamental group.

1.1.3 Geometric Points, Fibres, and Galois Covers

In parallel with the topological case, defining the étale fundamental group requires endowing both the base space and its covering spaces with base points. A *geometric point* \overline{y} of a scheme Y is a morphism

$$\overline{y}: \text{Spec}(\Omega) \longrightarrow Y,$$

where Ω is a separably closed field. This is equivalent to a choice of point y in the underlying topological space of Y together with an embedding of the residue field at y into a separably closed field. If $f: Y \rightarrow X$ is a morphism of schemes, we write $f(\overline{y})$ for the composition $f \circ \overline{y}$ — the image of \overline{y} in X .

⁵A proof of this can be found in [33, Prop. 5.3.1].

⁶This is proved in Conrad's notes [6, Thm. 1.1.7.2].

Now, if $\bar{x}: \text{Spec}(\Omega) \rightarrow X$ is a geometric point of X and $f: Y \rightarrow X$ is an étale cover, the *geometric fibre* of Y above \bar{x} is the set

$$F_{\bar{x}}(Y) := \{\bar{y}: \text{Spec}(\Omega) \rightarrow Y \mid f(\bar{y}) = \bar{x}\}.$$

- Example 1.1.3.** 1. A choice of geometric point for $\text{Spec}(K)$ amounts to a choice of embedding of K into the separably closed field Ω . The étale covers of $\text{Spec}(K)$ are given by inclusions of K into finite étale K -algebras. The elements of the geometric fibre of any such cover is therefore the collection of embeddings of its component finite separable extensions into Ω that are consistent with the choice of embeddings of K into Ω and the algebra.
2. A geometric point of $\mathbb{G}_{m,K} = \text{Spec}(K[x, y]/(xy - 1))$ is given by a choice of image for x in Ω^\times . With such a choice, the geometric fibre of a connected étale cover (as described in Example 1.1.1.3) consists, when K is algebraically closed, of points \bar{y} such that

$$\begin{array}{ccc} K[x, y]/(xy - 1) & \xrightarrow{\bar{y}} & \Omega \\ \uparrow [d] & \nearrow x \mapsto \omega & \\ K[x, y]/(xy - 1) & & \end{array}$$

commutes. Thus, \bar{y} maps x to a d^{th} root of ω . When K is not algebraically closed, the covering space includes a choice of finite separable extension F of K , and the points of the geometric fibre are a choice of d^{th} root of ω *and* a choice of embedding of F into Ω that respects the embedding of K into both fields.

Being a finite morphism, an étale cover necessarily has finite geometric fibres.

When the base scheme is connected, this finite cardinality must be identical across all geometric points. Even more can be said when the covering space X is also connected.

For an étale cover $f: Y \rightarrow X$, write $\text{Aut}_X(Y)$ for the set of isomorphisms φ of Y such that

$$\begin{array}{ccc} Y & \xleftarrow{\varphi} & Y \\ & \searrow f & \swarrow f \\ & X & \end{array}$$

commutes. When the covering map itself requires emphasis, we will use the alternate notation $\text{Aut}_X(f)$. The automorphism group acts on the geometric fibre of the cover via

$$\varphi \cdot \overline{y} := \varphi(\overline{y}),$$

and when Y is connected, the fact that any section maps a connected component of the base onto a connected component of the cover implies that any $\varphi \in \text{Aut}_X(Y)$ is actually entirely determined by the image of \overline{y} . From that same fact it also follows that the action of the automorphism group on the fibre has no fixed points when Y is connected⁷; thus, when X and Y are both connected, the size of the automorphism group is bounded by the size of the geometric fibre:

$$|\text{Aut}_X(Y)| \leq |F_{\overline{x}}(Y)|.$$

When $|\text{Aut}_X(Y)| = |F_{\overline{x}}(Y)|$, f is said to be a *Galois* étale cover, or simply a Galois cover.

Example 1.1.4. 1. As in Example 1.1.1.2, étale covers of the spectrum of a field

⁷These are proved as Corollaries 5.3.3 and 5.3.4 in [33].

K are all by spectra of finite étale K -algebras. Such a cover is Galois when all the constituent extensions are Galois, and is connected exactly when it is composed of a single finite separable extension.

2. All of the connected étale covers of $\mathbb{G}_{m,K}$ when K is algebraically closed described in Example 1.1.1.3,

$$\begin{aligned} [d]: K[x, y]/(xy - 1) &\longrightarrow K[x, y]/(xy - 1) \\ x &\longmapsto x^d \\ y &\longmapsto y^{-d}, \end{aligned}$$

for any $d \in \mathbb{Z}$, are Galois. The automorphism groups of these covers are

$$\mathrm{Aut}_{\mathbb{G}_{m,K}}([d]) \cong \mu_{d,K}.$$

When K is not algebraically closed the étale covers are

$$\begin{aligned} [d]_F: K[x, y]/(xy - 1) &\longrightarrow F[x, y]/(xy - 1) \\ K &\hookrightarrow F \\ x &\longmapsto x^d \\ y &\longmapsto y^{-d} \end{aligned}$$

for any integer d and finite separable extension $K \hookrightarrow F$. In this case the covering group is

$$\mathrm{Aut}_{\mathbb{G}_{m,K}}([d]_F) \cong \mu_{d,F} \times \mathrm{Gal}(F/K);$$

therefore, the cover is Galois precisely when F is a Galois extension containing all d^{th} roots of unity.

1.1.4 The Fundamental Group

Like the fundamental group familiar from topology, the étale fundamental group is meant to classify the étale covers in the sense that the automorphism groups of the covers appear as finite quotients of the group, with the relationship between quotient groups reflecting a relationship between the corresponding covers.

To allow this idea to lead us toward the definition, suppose we have an étale cover $f: Y \rightarrow X$ and another connected étale cover $g: Z \rightarrow X$ such that

$$\begin{array}{ccc} Y & \xrightarrow{h} & Z \\ & \searrow f & \swarrow g \\ & X & \end{array}$$

commutes. Then, as one can find proved in [33, Prop. 5.3.8], the factorizing map h is also an étale cover exactly when there exists a normal subgroup H of $\text{Aut}_X(Y)$ such that $\text{Aut}_X(Z) = \text{Aut}_X(Y)/H$. So, just as in the topological case, there is a bijection between étale covers factoring f and subgroups of the automorphism group. In the process this also shows that every commuting triangle

$$\begin{array}{ccc} Y & \xrightarrow{h} & Z \\ & \searrow f & \swarrow g \\ & X & \end{array}$$

of étale covers gives rise to a homomorphism $\text{Aut}_X(Y) \rightarrow \text{Aut}_X(Z)$ that is, in fact, a quotient map. It has a concrete description that we will make use of in later chapters:

For every $\varphi \in \text{Aut}_X(Y)$ there is exactly one $\psi \in \text{Aut}_X(Z)$ such that

$$\psi \circ h = h \circ \varphi; \quad (1.1.4.1)$$

the quotient map is defined by $\varphi \mapsto \psi$.

All this invites imposing a partial order on $\text{Fet}(X)$ according to the maps between covers and endowing it (along with the accompanying category of automorphism groups) with the structure of a projective system. Without base points this is impossible since any morphism of covering spaces $Y \rightarrow Z$ need not be unique. But if each covering space Y of X is endowed with a base point $\bar{y} \in F_{\bar{x}}(Y)$, there is then at most one morphism between covering spaces that also respects the base points. This provides both the category of étale covers and the category of the covering groups of étale covers with a partial order, giving them the structure of projective systems, which allows the following definition.

Definition 1.1.5. The *étale fundamental group* of a connected scheme X with geometric point \bar{x} is

$$\pi_1(X, \bar{x}) := \varprojlim_{\bar{Y}} \text{Aut}_X(Y)^{\text{op}},$$

where the limit is taken over all connected Galois covers of X . We assume the finite groups appearing in the limit are discretely topologized, so that $\pi_1(X, \bar{x})$ carries the profinite topology.

This definition is not actually the one originally given in SGA [13, Exp. V, §7]. There, after much heaving of categorical machinery into place, the fundamental group is defined as the automorphism group of the *fibre functor* associated to the chosen

geometric point \bar{x} ,

$$\begin{aligned} F_{\bar{x}}: \text{Fet}(X) &\longrightarrow \text{Ens} \\ Y &\longmapsto F_{\bar{x}}(Y). \end{aligned}$$

Our definition is equivalent, and ultimately more agreeable to work with for being more concrete. The connection between the two definitions can be seen by considering once again the topological case, where (under the appropriate conditions) the analogous fibre functor is represented by a single covering space, the universal cover. The topological fundamental group is isomorphic to the automorphism group of that space. Precisely the same thing would be true in the étale case if the projective limit of the Galois covering spaces of X existed in $\text{Fet}(X)$. It does not, and while one might remedy this incongruity between the two situations by enlarging the category from which covers of X can be drawn, in the literature it is more common let it be, defer taking limits until after passing to automorphism groups (where the opposite groups are used to ensure that the resulting group retains a left action on the fibres of the étale covers of X) to obtain the étale fundamental group, and say that instead of being representable the fibre functor is *pro-representable* in the étale context.

Viewing the fundamental group as the automorphism group of the fibre functor does, however, make it easier to see certain key facts; most importantly, that the fundamental group is functorial. Let X and Y be connected schemes with geometric points \bar{x} and \bar{y} , and $\alpha: X \rightarrow Y$ a scheme morphism such that $\alpha(\bar{x}) = \bar{y}$. Base change by α gives a functor $\mathcal{B}_\alpha: \text{Fet}(Y) \rightarrow \text{Fet}(X)$, mapping any étale cover U of Y to

$X \times_Y U$, as in the Cartesian diagram

$$\begin{array}{ccc} X \times_Y U & \xrightarrow{\alpha^U} & U \\ \downarrow & & \downarrow \\ X & \xrightarrow{\alpha} & Y. \end{array}$$

That α respects the geometric points means that α^U carries any $\bar{u} \in F_{\bar{x}}(X \times_Y U)$ to a point in $F_{\bar{y}}(U)$. Conversely, any point in $F_{\bar{y}}(U)$ defines, via the universal property, a point in $F_{\bar{x}}(X \times_Y U)$. Therefore, $F_{\bar{x}} \circ \mathcal{B}_\alpha = F_{\bar{y}}$. So any automorphism of $F_{\bar{x}}$ will give rise to an automorphism of $F_{\bar{y}}$ via \mathcal{B}_α , giving a map

$$\pi_1(\alpha): \pi_1(X, \bar{x}) \longrightarrow \pi_1(Y, \bar{y}).$$

This map will be the true workhorse of Chapters 3 and 4, so we need to characterize it explicitly. We can do this by returning to the view of the fundamental group as an inverse limit.

Let $(\varphi_U) \in \pi_1(X, \bar{x})$, so that U ranges over the connected Galois covers of X . Then the component of $\pi_1(\alpha)((\varphi_U))$ indexed by a connected Galois cover V of Y is the unique covering transformation ψ_V of V satisfying

$$f_V \circ \varphi_{X \times_Y V} = \psi_V \circ f_V \tag{1.1.5.1}$$

as in the commuting diagram

$$\begin{array}{ccccccc}
 X \times_Y V & \xrightarrow{\varphi_{X \times_Y V}} & X \times_Y V & \xrightarrow{\alpha^V} & V & \xrightarrow[\exists!]{\psi_V} & V \\
 & \searrow & \downarrow & & \downarrow & \swarrow & \\
 & & X & \xrightarrow{\alpha} & Y & &
 \end{array} \tag{1.1.5.2}$$

Such a transformation must exist because V is a Galois cover.

As a final note, this functoriality property implies that if \bar{x} and \bar{x}' are two geometric points of X , then there is an isomorphism

$$\pi_1(X, \bar{x}) \xrightarrow{\sim} \pi_1(X, \bar{x}')$$

that is unique up to an inner automorphism⁸. The abelianization of $\pi_1(X, \bar{x})$ is therefore independent of the base point chosen.

Example 1.1.6.

1. $\text{Spec}(K)$. The previous round of examples described the connected étale covers of $\text{Spec}(K)$ as the spectra of finite separable extensions of K . Thus,

$$\pi_1(\text{Spec}(K), \bar{x}) = \varprojlim_{L/K \text{ fin. sep.}} \text{Gal}(L/K) = \text{Gal}(K^{\text{sep}}/K).$$

2. $\mathbb{G}_{m,K}$. Let $e: \text{Spec}(K) \rightarrow \mathbb{G}_{m,K}$ be the identity of $\mathbb{G}_{m,K}$ and $\bar{e}: \text{Spec}(\bar{K}) \rightarrow \text{Spec}(K) \rightarrow \mathbb{G}_{m,K}$ and take this to be our geometric point. When K is alge-

⁸See [33, Cor. 5.5.2] and the subsequent remark for proof.

braically closed, by Example 1.1.4, we have

$$\begin{aligned}\pi_1(\mathbb{G}_{m,K}, \bar{e}) &= \varprojlim_{n \in \mathbb{N}} \mu_n \\ &\approx \varprojlim_{n \in \mathbb{N}} \mathbb{Z}/n\mathbb{Z} \\ &= \widehat{\mathbb{Z}}.\end{aligned}$$

When K is not algebraically closed, we have instead,

$$\begin{aligned}\pi_1(\mathbb{G}_{m,K}) &= \varprojlim_{n \in \mathbb{N}} \mu_n \times \text{Gal}(K(\mu_n)/K) \\ &\approx \varprojlim_{n \in \mathbb{N}} \mathbb{Z}/n\mathbb{Z} \times \varprojlim_{n \in \mathbb{N}} \text{Gal}(K(\mu_n)/K) \\ &= \widehat{\mathbb{Z}} \times \text{Gal}((K^{\text{sep}})^{\text{ab}}/K)\end{aligned}$$

3. \mathbb{A}_K^1 . From Example 1.1.4.3, when K is separably closed

$$\pi_1(\mathbb{A}_K^1, \bar{x}) = 0.$$

4. $\text{Spec}(\mathbb{Z})$. Likewise,

$$\pi_1(\text{Spec}(\mathbb{Z}), \bar{x}) = 0.$$

Of course, completely enumerating all connected Galois covers of an arbitrary connected scheme in order to calculate its fundamental group is not a trivial matter. Fortunately, it is sometimes possible to use knowledge of topological fundamental groups of complex manifolds to determine étale fundamental groups, courtesy of the next two propositions.

To any scheme X of finite type over \mathbb{C} it is possible to associate a complex analytic

space X^{an} that, when X is smooth, is a complex manifold (for details, see SGA I [13, Exposé XII]). The obvious question then is what relation the topological fundamental group of X^{an} has to the étale fundamental group of X .

Proposition 1.1.7 (SGA I [13] Exposé XII, Cor. 5.2). *Let X be a connected scheme of finite type over \mathbb{C} with analytification X^{an} . For every \mathbb{C} -point $\bar{x}: \text{Spec}(\mathbb{C}) \rightarrow X$ there is an isomorphism*

$$\pi_1^{\text{top}}(\widehat{X^{\text{an}}}, \bar{x}) \xrightarrow{\sim} \pi_1(X, \bar{x}),$$

where the hat denotes the profinite completion of the topological fundamental group of X^{an} .

Combined with the following proposition, this allows us to use knowledge of topological fundamental groups of complex manifolds to determine the étale fundamental group of schemes over $\overline{\mathbb{Q}}_\ell$, since there exists a (nonunique) isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$.

Proposition 1.1.8 (Szamuely, Prop. 5.6.7). *Let $K \hookrightarrow F$ be an extension of algebraically closed fields, X a proper integral scheme over K , \bar{x} a geometric point of X , and X_F the extension of X to F with corresponding geometric point \bar{x}_F . Then, the functorial map*

$$\pi_1(X_F, \bar{x}_F) \longrightarrow \pi_1(X, \bar{x})$$

coming from the projection $X_F \rightarrow X$ is an isomorphism.

1.1.5 Finite Locally Constant Sheaves and the Fundamental Group

The action of $\pi_1(X, \bar{x})$ on the geometric fibre of any connected étale cover Y comes from an action on the cover Y itself. Any element of $\gamma \in \pi_1(X, \bar{x})$ can be regarded as a compatible tuple of covering transformations of étale covers of X . The fundamental

group acts on $F_{\bar{x}}(Y)$ through the covering group's action on the fibre:

$$\gamma \cdot \bar{y} := \varphi_Y \cdot \bar{y} = \varphi_Y(\bar{y}) = \varphi_Y \circ \bar{y}.$$

This observation leads to the fundamental result in the theory of the étale fundamental group, and the first step toward the equivalence between local systems and representations of the fundamental group: Let X be a connected scheme over a field K with geometric point \bar{x} . The fibre functor $F_{\bar{x}}$ defines an equivalence between the category of étale covers of X and the category $\text{Ens}(\pi_1(X, \bar{x}))$ of finite sets equipped with a continuous left action of $\pi_1(X, \bar{x})$. Connected covers correspond to sets with transitive action, and Galois covers to finite quotients of $\pi_1(X, \bar{x})$.⁹ Here, ‘continuous’ means with respect to the profinite topology of the fundamental group. Combined with Proposition 1.1.2 in the case of a connected scheme X , we obtain an equivalence between $\text{Flc}(X)$ and $\text{Ens}(\pi_1(X, \bar{x}))$. The composite functor carries a finite locally constant sheaf to its stalk at \bar{x} .

This fact can be refined by restricting attention to the group objects in each of the categories, obtaining the next step in the road toward the equivalence between local systems and fundamental group representations.

Proposition 1.1.9. *Let X be a connected scheme over a field K and $\bar{x}: \text{Spec}(\bar{K}) \rightarrow X$ a geometric point of X . The functor*

$$\mathcal{F} \longmapsto \mathcal{F}_{\bar{x}}$$

defines an equivalence of categories between finite locally constant sheaves of finite

⁹A proof can be found in Szamuely [33, Thm. 5.4.2].

abelian groups and finite abelian groups equipped with a continuous left action of $\pi_1(X, \bar{x})$.

1.1.6 The Homotopy Exact Sequence

Finally, we end this section by recalling a crucial exact sequence associated to the étale fundamental group. It will allow us to easily define and keep track of Galois actions on sheaves and local systems.

Let X be a quasi-compact and geometrically integral scheme over a field K with algebraic closure \bar{K} and separable closure K^{sep} . Let \bar{X} denote the base extension of X to $\text{Spec}(K^{\text{sep}})$, and choose a geometric point $\bar{x}: \text{Spec}(\bar{K}) \rightarrow \bar{X}$ for \bar{X} and write \bar{x} for its image in X under the canonical projection. With all this, the sequence, called the *homotopy exact sequence*,

$$1 \rightarrow \pi_1(\bar{X}, \bar{x}) \rightarrow \pi_1(X, \bar{x}) \rightarrow \text{Gal}(K^{\text{sep}}/K) \rightarrow 1$$

of profinite groups, where the inner two maps are functorially induced by the maps $\bar{X} \rightarrow X$ and $X \rightarrow \text{Spec}(K)$, is exact. On its own this provides a means of determining the fundamental group of \mathbb{A}_K^2 where K is not algebraically closed. Since, for a separable algebraic closure \bar{K} of K , $\pi_1(\mathbb{A}_{\bar{K}}^2, \bar{x}) = 0$, it immediately follows from the homotopy exact sequence that $\pi_1(\mathbb{A}_K^2, \bar{x}) = \text{Gal}(\bar{K}/K)$.

If the scheme X is defined over a field K and has a K -rational point s such that

$$\begin{array}{ccc} & & X \\ & \nearrow s & \uparrow \bar{x} \\ \mathrm{Spec}(K) & & \mathrm{Spec}(\bar{K}) \\ & \nwarrow & \end{array}$$

commutes, the functorial map $\pi_1(s): \mathrm{Gal}(K^{\mathrm{sep}}/K) \rightarrow X$ necessarily gives a splitting of the homotopy exact sequence, allowing us to write

$$\pi_1(X, \bar{x}) \cong \pi_1(\bar{X}, \bar{x}) \rtimes \mathrm{Gal}(K^{\mathrm{sep}}/K).$$

Using the identity e , this fact gives a different way of seeing that the fundamental group of $\mathbb{G}_{m,K}$ is as described in Example 1.1.6.2.

1.2 ℓ - and π -adic Sheaves

This section is devoted to recalling the definition and basic facts about the category of π -adic sheaves, which are fundamental to the definition of $\overline{\mathbb{Q}}_\ell$ -local systems. Most important is the equivalence between the category of $\overline{\mathbb{Q}}_\ell$ -local systems and representations of the étale fundamental group. Whenever possible, we will make use of this equivalence to avoid the cumbersome work of dealing with the local systems themselves.

The definitive source for ℓ -adic sheaves remains Exposés V and VI of SGA 5 [3]. Freitag and Kiehl's book [12] provides most of the information here in greater depth

for the case of ℓ -adic sheaves, which π -adic sheaves generalize in a straightforward way. All of the results there remain valid for π -adic sheaves. The book by Kiehl and Weissauer [17] discusses the extension of some deeper results to $\overline{\mathbb{Q}}_\ell$ -sheaves. Another good source is section 1.4 of Conrad's notes, *Étale Cohomology* [6].

A projective system of sheaves on the étale site of X

$$\cdots \rightarrow \mathcal{F}_{n+1} \rightarrow \mathcal{F}_n \rightarrow \mathcal{F}_{n-1} \rightarrow \cdots$$

will be written $\mathcal{F} = (\mathcal{F}_n)_{n \in \mathbb{Z}}$ or just (\mathcal{F}_n) , and a morphism between two systems $f: (\mathcal{F}_n) \rightarrow (\mathcal{G}_n)$

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathcal{F}_{n+1} & \longrightarrow & \mathcal{F}_n & \longrightarrow & \mathcal{F}_{n-1} \longrightarrow \cdots \\ & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} \\ \cdots & \longrightarrow & \mathcal{G}_{n+1} & \longrightarrow & \mathcal{G}_n & \longrightarrow & \mathcal{G}_{n-1} \longrightarrow \cdots \end{array}$$

similarly as $f = (f_n)$ when the maps between individual components need emphasis. Finally, let E be a finite extension of the ℓ -adic numbers, \mathbb{Q}_ℓ , with valuation ring \mathcal{O}_E and uniformizer π .

Throughout this section let X be a Noetherian scheme on which the prime ℓ is invertible (meaning that ℓ maps to an invertible element in every ring of sections of the structure sheaf of X). We make the Noetherian requirement because the definition of π -adic sheaves uses constructibility, which is a simpler concept on Noetherian schemes. An étale sheaf \mathcal{F} on a Noetherian scheme X is *constructible* if X can be written as a finite union of locally closed subschemes such that the restriction of \mathcal{F} to each locally closed stratum is finite locally constant.

1.2.1 π -adic Sheaves

A π -adic sheaf on X is a projective system of étale sheaves of \mathcal{O}_E -modules $(\mathcal{F}_n)_{n \in \mathbb{Z}}$ satisfying

- (i) \mathcal{F}_n is constructible for all n ,
- (ii) $\mathcal{F}_n = 0$ for $n < 0$,
- (iii) $\pi^{n+1}\mathcal{F}_n = 0$ for $n \geq 0$,
- (iv) $\mathcal{F}_{n+1} \otimes_{(\mathcal{O}_E/\pi^{n+2}\mathcal{O}_E)} (\mathcal{O}_E/\pi^{n+1}\mathcal{O}_E) \cong \mathcal{F}_n$ for all $n \geq 0$.

When $\mathcal{O}_E = \mathbb{Z}_\ell$ and $\pi = \ell$, (\mathcal{F}_n) is called an ℓ -adic sheaf. A π -adic sheaf is *locally constant* if \mathcal{F}_n is locally constant for every n . If \bar{x} is a geometric point of X , the *stalk* of $\mathcal{F} = (\mathcal{F}_n)$ at \bar{x} is

$$\mathcal{F}_{\bar{x}} := \varprojlim_{n \in \mathbb{N}} (\mathcal{F}_n)_{\bar{x}}.$$

Despite the name, it is not true that there exists an étale covering on which a locally constant π -adic sheaf becomes constant (*i.e.*, all of the constituent sheaves \mathcal{F}_n become constant simultaneously) after restriction. An easy instance where this fails is the Tate twist $\mathbb{Z}_\ell(1)$ described in Example 1.2.1. Therefore, the term “locally constant π -adic sheaf” is rarely used in the literature. Unfortunately, the terminology that is in use is confusingly varied. In SGA, locally constant ℓ -adic sheaves are called *constant tordeu*. Now they are more commonly called *lisse* or *smooth*. We will call them smooth.

There is also the completely analogous concept of a π -adic system of \mathcal{O}_E -modules $M = (M_n)$ for which the defining properties are

- (i) M_n is a module of finite length for all n ,
- (ii) $M_n = 0$ for $n < 0$,

- (iii) $\pi^{n+1}M_n = 0$ for $n \geq 0$,
- (iv) $M_{n+1} \otimes_{\mathcal{O}/\pi^{n+2}\mathcal{O}} (\mathcal{O}_E/\pi^{n+1}\mathcal{O}_E) \cong M_n$ for all $n \geq 0$.

Transporting entire π -adic sheaves using the equivalence in Proposition 1.1.9 results in a π -adic system of \mathcal{O}_E -modules.

Example 1.2.1. (i) The constant sheaf ℓ -adic sheaf $(\mathbb{Z}_\ell)_X$ is

$$\cdots \rightarrow (\mathbb{Z}/\ell^2\mathbb{Z})_X \rightarrow (\mathbb{Z}/\ell\mathbb{Z})_X \rightarrow 0_X.$$

More generally, the constant sheaf $(\mathcal{O}_E)_X$ (not to be confused with the structure sheaf \mathcal{O}_X) is similarly a π -adic sheaf.

- (ii) Let $(\mu_{\ell^n})_X$ be the sheaf of ℓ^n -th roots of unity on X (that is, the kernel of the ℓ^n power map on \mathcal{O}_X^* , the sheaf of units of the structure sheaf of X). Then, the projective system of these sheaves, with $(\mu_{\ell^{n+1}})_X \rightarrow (\mu_{\ell^n})_X$ the ℓ th power map, is an ℓ -adic sheaf, the *Tate twist* of the constant sheaf, $(\mathbb{Z}_\ell(1))_X$. More generally, the m th Tate twist, $(\mathbb{Z}_\ell(m))_X := ((\mu_{\ell^n})_X^{\otimes m})$, is an ℓ -adic sheaf.

The Tate twist provides an easy example of the “failure of local constancy” noted above. Suppose $X = \operatorname{Spec}(K)$, K a field. Then, the sheaves $(\mu_{\ell^n})_X$ would only become constant simultaneously over an étale neighbourhood defined by a finite extension of K containing all ℓ -power roots of unity. Of course, no such field exists. But each component sheaf does become constant individually over an appropriately large cyclotomic extension of K , so $(\mathbb{Z}_\ell(1))_X$ is a locally constant or, as we prefer, smooth, π -adic sheaf.

1.2.2 E -Local Systems

Smooth π -adic sheaves give rise to local systems after being embedded in the category of sheaves of E -vector spaces. To define those, we need to introduce the Artin-Rees, or A-R, category first. This category arises out of an effort to construct a category that carries over the abelian category structures and operations (kernels, cokernels, Hom, and so on) of sheaves of modules to the projective system category by applying them term by term. As defined, this isn't possible in the category of π -adic sheaves itself, since, for example, the projective system of kernels $(\ker(f_n))$ of a morphism

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathcal{F}_{n+1} & \longrightarrow & \mathcal{F}_n & \longrightarrow & \mathcal{F}_{n-1} \longrightarrow \cdots \\ & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} \\ \cdots & \longrightarrow & \mathcal{G}_{n+1} & \longrightarrow & \mathcal{G}_n & \longrightarrow & \mathcal{G}_{n-1} \longrightarrow \cdots \end{array}$$

does not necessarily satisfy condition (iii) of the definition. The expanded A-R category addresses this¹⁰.

Write $\mathcal{F}[r]$ for the projective system $\mathcal{F} = (\mathcal{F}_n)_{n \in \mathbb{N}}$ shifted r degrees; *i.e.*, $\mathcal{F}[r] = (\mathcal{F}_{n+r})_{n \in \mathbb{Z}}$. The objects of the A-R category of sheaves are all projective systems (\mathcal{F}_n) whose component sheaves are π -torsion (*i.e.*, every section is annihilated by some power of π). The morphisms in the category are given by

$$\mathrm{Hom}_{A-R}(\mathcal{F}, \mathcal{G}) := \varprojlim_r \mathrm{Hom}(\mathcal{F}[r], \mathcal{G}).$$

Any projective system in the A-R category that is isomorphic to an π -adic sheaf is called *A-R π -adic*. With this, we can begin to develop the notion of a local system.

¹⁰Conrad's notes [6] are more expansive on this topic if the reader is interested.

The category of *sheaves of E -vector spaces* on X has the same objects as the Artin-Rees category of π -adic sheaves, written $\mathcal{F} \otimes E$ to distinguish them. For any two sheaves of E -vector spaces $\mathcal{F} \otimes E$ and $\mathcal{G} \otimes E$,

$$\mathrm{Hom}(\mathcal{F} \otimes E, \mathcal{G} \otimes E) := \mathrm{Hom}_{A-R}(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{O}_E} E.$$

Local systems are special objects within this category. Specifically, an *E -local system* \mathcal{L} on X is a sheaf of E -vector spaces isomorphic to $\mathcal{F} \otimes E$, where \mathcal{F} is a smooth π -adic sheaf. Smoothness means that if X is connected and \bar{x} and \bar{x}' are any two geometric points of X , the stalks of \mathcal{L} at \bar{x} and \bar{x}' are isomorphic. In particular, they are E -vector spaces of the same dimension, and this common dimension is the *rank* of \mathcal{L} .

In the previous section we noted that there was an equivalence between finite locally constant sheaves of abelian groups and finite abelian groups with a continuous action of the étale fundamental group. That equivalence can be extended to the objects now at hand. First, by applying Proposition 1.1.9 to each component sheaf, we see that a smooth π -adic sheaf is equivalent to a π -adic system of \mathcal{O}_E -modules

$$\cdots \rightarrow M_{n+1} \rightarrow M_n \rightarrow M_{n-1} \rightarrow \cdots,$$

where each M_i is an $\mathcal{O}_E/\pi^i \mathcal{O}_E$ -module equipped with a continuous action of $\pi_1(X, \bar{x})$. In SGA 5 it is shown that the inverse limit functor defines an equivalence between the category of π -adic modules and the category of finitely generated \mathcal{O}_E -modules with continuous fundamental group action. By composition, we obtain:

Proposition 1.2.2. *[SGA 5 [3], Exp. VI, Lemme 1.2.4.2] The category of smooth*

π -adic sheaves on a scheme X is equivalent to the category of finitely generated \mathcal{O}_E -modules equipped with a continuous action of the étale fundamental group of X .

Finally, the obvious functor from the category of π -adic sheaves to the category of sheaves of E -vector spaces is mirrored in the module category by the “tensor with E ” functor

$$M \longmapsto M \otimes_{\mathcal{O}_E} E$$

into the category of E vector spaces, providing an equivalence between the category of E -local systems on a connected scheme X with the category of E -valued representations of $\pi_1(X, \bar{x})$ ¹¹. Under this equivalence, the rank of a local system is equal to the dimension of the corresponding representation.

1.2.3 $\overline{\mathbb{Q}}_\ell$ -Local Systems

Finally, we will define the category of $\overline{\mathbb{Q}}_\ell$ -local systems, which is built up as a direct limit of the categories of E -local systems. To see how¹², note that for any finite extension of local fields, $E \subseteq L$, with rings of integers \mathcal{O}_E and \mathcal{O}_L respectively, and uniformizers π_E and π_L such that $\pi_L^e = \pi_E$, where e is the ramification index of the extension, there is a functor from the category of π_E -adic sheaves of X to that of π_L -adic sheaves on X defined by

$$\mathcal{F} = (\mathcal{F}_n) \longmapsto \overline{\mathcal{F}} = (\overline{\mathcal{F}}_n),$$

¹¹Conrad sketches a proof in [6, Thm. 1.4.5.4]

¹²Conrad [6] takes a different approach to this construction; see Definition 1.4.5.6.

where

$$\overline{\mathcal{F}}_{ne} := \mathcal{F}_n \otimes_{(\mathcal{O}_E/\pi_E^{n+1}\mathcal{O}_E)} \mathcal{O}_L/\pi_L^{(n+1)e}$$

for all n and for $1 \leq i < e$,

$$\overline{\mathcal{F}}_{ne-i} := \mathcal{F}_n \otimes_{(\mathcal{O}_E/\pi_E^{n+1}\mathcal{O}_E)} \mathcal{O}_L/\pi_L^{(n+1)e-i} \mathcal{O}_L.$$

This functor can be extended to one from the category of sheaves of E -vector spaces on X to the category of L -vector spaces by

$$\mathcal{F} \otimes E \longmapsto \overline{\mathcal{F}} \otimes L.$$

From this, we get a directed system of categories, the limit of which is the category of sheaves of $\overline{\mathbb{Q}}_\ell$ -vector spaces. Every object $\mathcal{F} \otimes \overline{\mathbb{Q}}_\ell$ in the category can be represented by a sheaf of E -vector spaces,

$$\mathcal{F} \otimes E,$$

and if there exists such a representative that is an E -local system, so that \mathcal{F} is a smooth π_E -adic sheaf, then we say that $\mathcal{F} \otimes \overline{\mathbb{Q}}_\ell$ is a $\overline{\mathbb{Q}}_\ell$ -local system.

As one would expect, the equivalence for E -local systems extends to $\overline{\mathbb{Q}}_\ell$ -local systems.

Proposition 1.2.3. *The category of sheaves of $\overline{\mathbb{Q}}_\ell$ -local systems (or, alternately, ℓ -adic local systems) on a connected scheme X is equivalent to the category of finite dimensional $\overline{\mathbb{Q}}_\ell$ representations of $\pi_1(X, \overline{x})$. The rank of a local system under this equivalence is equal to the dimension of the corresponding representation.*

Conrad's notes give a proof [6, Thm. 1.4.5.7].

1.2.4 Proper and Smooth Base Change

Before closing this chapter, we state two important theorems for reference. These are the base change theorems for proper and smooth morphisms, originally proved in SGA for torsion sheaves, which are sheaves of groups whose every group of sections is a torsion group. Both theorems extend from that case to the case of π -adic sheaves and $\overline{\mathbb{Q}_\ell}$ -local systems.

Theorem 1.2.4 (Proper Base Change, SGA 4(III) [2], Exposé XII, 5.1(i)). *Let X and Y be schemes and $f: X \rightarrow Y$ a proper morphism. Let*

$$\begin{array}{ccc} X \times_Y W & \xrightarrow{f'} & W \\ g' \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

be a Cartesian diagram. Then, for every torsion sheaf (or π -adic sheaf, or $\overline{\mathbb{Q}_\ell}$ -sheaf) \mathcal{F} on X , the base change isomorphism

$$g^*(R^i f_* \mathcal{F}) \xrightarrow{\sim} R^i f'_*(g'^* \mathcal{F})$$

is an isomorphism.

Theorem 1.2.5 (Smooth Base Change, SGA 4(III) [2], Exposé XVI). *Suppose*

$$\begin{array}{ccc} X \times_Y W & \xrightarrow{f'} & W \\ g' \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

is a Cartesian diagram of Noetherian schemes. If g is a smooth morphism and \mathcal{F}

is a torsion sheaf (π -adic or $\overline{\mathbb{Q}_\ell}$ -sheaf with torsion components) whose sections have order relatively prime to the residue characteristic of Y , then the base change homomorphism

$$g^*(R^i f_* \mathcal{F}) \longrightarrow R^i f'_*(g'^* \mathcal{F})$$

is an isomorphism.

Chapter 2

Galois Actions

In SGA 7(II) [1] Exposé XII, Deligne introduced the notion of a “Galois sheaf”¹ on a scheme X over a field K , and proved that the category of Galois sheaves on the extension $\overline{X} := X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})$ to a separable closure \overline{K} of K is equivalent to the category of sheaves on X . There are indications in the literature that the same kind of correspondence exists more generally. In particular, the remark following Proposition 5.1.2 in [4] states that it is true of complexes belonging to the bounded derived category of ℓ -adic sheaves. However, no indication of a proof is given there, nor, it seems, does one exist anywhere else. Since the results of Chapter 4 are formulated around the existence of this kind of correspondence, we give a proof here.

In this chapter we show that the sort of Galois action Deligne describes can easily be extended to π -adic sheaves and $\overline{\mathbb{Q}}_\ell$ -local systems. Less trivially, in the case of $\overline{\mathbb{Q}}_\ell$ -local systems, it turns out that a splitting of the homotopy exact sequence (as in Section 1.1.6) allows us to naturally view the data of a Galois action in Deligne’s sense as a part of the representation of the étale fundamental group equivalent to the local system. This gives a means of extending the equivalence proved in SGA for torsion sheaves (see Proposition 2.1.3) to a class of $\overline{\mathbb{Q}}_\ell$ -local systems that includes those we define in Chapter 3. And although the approach taken here may not establish this

¹Deligne does not officially define any objects with this name however since, as noted below, the notion is slightly broader. In light of the actual use he puts them to, this seems a reasonable name.

fact in its fullest generality, it does allow us to pursue our ongoing policy of working with fundamental group representations rather than the $\overline{\mathbb{Q}}_\ell$ -local systems to which they correspond.

We begin with a recollection of Deligne’s original Galois sheaf concept.

2.1 Deligne’s Galois Sheaves

As mentioned, all of the material here first appeared in SGA 7(II) [1] Exposé XII, although the presentation here is more detailed in certain respects so that this notion of a Galois action on a sheaf can be clearly compared with the Galois action implicitly contained in the representation of a local system on X as a representation of the fundamental group of X when X has a K -rational point. A thorough review of the details is also advisable because, despite the name, Deligne’s notion of a Galois action on a sheaf extends to any profinite group equipped with a continuous homomorphism to the Galois group in question, and this provides a crucial structural aspect of the nearby cycles functor, which is vital to the results of Chapter 4.

Let X be a scheme over a field K with separable closure \overline{K} , and let $\overline{X} := X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})$. To begin, we allow G to be a profinite group equipped with a continuous homomorphism

$$\epsilon: G \longrightarrow \mathrm{Gal}(\overline{K}/K).$$

The Galois group naturally acts on \overline{X} via its action on \overline{K} , and we will denote the

induced action of $\gamma \in \text{Gal}(\overline{K}/K)$ by γ_X . This is visualized in the diagram

$$\begin{array}{ccccc} \overline{X} & \xrightarrow[\sim]{\gamma_X} & \overline{X} & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \text{Spec}(\overline{K}) & \xrightarrow[\text{Spec}(\gamma)]{\sim} & \text{Spec}(\overline{K}) & \longrightarrow & \text{Spec}(K) \end{array}$$

where all squares in sight are Cartesian. The group G therefore acts on \overline{X} via the homomorphism ϵ .

Definition 2.1.1. Let \mathcal{F} be an étale sheaf of sets on \overline{X} . An *action of G on \mathcal{F} compatible with the action of G on \overline{X}* is a system of isomorphisms, one for each $g \in G$,

$$\mu(g): \epsilon(g)_* \mathcal{F} \xrightarrow{\sim} \mathcal{F}$$

that satisfy, for any $h \in G$, $\mu(g \circ h) = \mu(g) \circ \epsilon(g)_*(\mu(h))$.

This idea takes the shape of a recognizable group action when we consider how the $\mu(g)$ act on sections of \mathcal{F} over an étale scheme \overline{U} arising via base change from an étale scheme U over X . In that case, the pullback of the diagram

$$\begin{array}{ccc} & \overline{U} & \\ & \downarrow & \\ \overline{X} & \xrightarrow{\gamma_X} & \overline{X} \end{array}$$

is again \overline{U} and so

$$(\mu(g))(\overline{U}): \epsilon(g)_* \mathcal{F}(\overline{U}) = \mathcal{F}(\overline{U}) \xrightarrow{\sim} \mathcal{F}(\overline{U})$$

is a permutation of the sections of \mathcal{F} over \overline{U} .

Definition 2.1.2. A profinite group G *acts continuously* on \mathcal{F} if, for any U quasi-compact and étale over X , G acts continuously on the discrete set $\mathcal{F}(\overline{U})$.

The fundamental example of such an action arises like so: If, instead of \overline{X} , \mathcal{F} is a sheaf of sets on X , and $\overline{\mathcal{F}}$ is the inverse image of \mathcal{F} under the projection $\overline{X} \rightarrow X$, then $\overline{\mathcal{F}}$ naturally carries an action of $\text{Gal}(\overline{K}/K)$ that is compatible with the action on \overline{X} . For every $\gamma \in \text{Gal}(\overline{K}/K)$, we have the Cartesian square

$$\begin{array}{ccc} \overline{U} \times_{\overline{X}} \overline{X} & \xrightarrow{\sim} & \overline{U} \\ \downarrow & & \downarrow \\ \overline{X} & \xrightarrow{\gamma_X} & \overline{X}. \end{array}$$

The isomorphism $\mu(\gamma): (\gamma_X)_* \mathcal{F}(\overline{U}) \xrightarrow{\sim} \mathcal{F}(\overline{U})$ is simply the restriction map $\overline{\mathcal{F}}(\overline{U} \times_{\overline{X}} \overline{X} \rightarrow \overline{U})$. This action is continuous in the sense above, which can be proved by considering the situation at the finite level: let L be a finite extension of K and write X_L for the base extension of X to L and \mathcal{F}_L for the inverse image of \mathcal{F} under $X_L \rightarrow X$. Then $\overline{\mathcal{F}} = \varprojlim_L \mathcal{F}_L$ and $\text{Gal}(\overline{K}/K)$ acts on \mathcal{F}_L , through the finite quotient $\text{Gal}(L/K)$, so that the action is necessarily continuous. Since the morphisms giving the action on $\overline{\mathcal{F}}$ are inverse limits of the morphisms at the finite level, it is also continuous.

There is therefore a functor from sheaves on X to Galois sheaves on \overline{X} , which is in fact one half of an equivalence, as Deligne proved.

Proposition 2.1.3. [SGA 7(II) [1] Exposé XII, Rappel 1.1.3(i)] *The category of sheaves of sets on the étale site of X is equivalent to the category of sheaves on \overline{X} equipped with a continuous action of $\text{Gal}(\overline{K}/K)$ compatible with the action on \overline{X} .*

The quasi-inverse of the functor described above is given, predictably, by invari-

ants. Let π be the projection $\overline{X} \rightarrow X$. The pushforward $\pi_* \overline{\mathcal{F}}$ inherits an action of $\text{Gal}(\overline{K}/K)$ from $\overline{\mathcal{F}}$ that is a straightforward action on the set of sections over every étale scheme over X , as described above. Therefore, it makes sense to take invariants under this action, and we define

$$\overline{\mathcal{F}}^{\text{Gal}(\overline{K}/K)} := (\pi_* \overline{\mathcal{F}})^{\text{Gal}(\overline{K}/K)}.$$

2.2 Galois Actions on π -adic Sheaves and Local Systems

Definition 2.1.1 can be naturally extended to the category of π -adic sheaves by taking the morphisms $\mu(g)$ to be morphisms in that category, with the pushforward by $\epsilon(g)$ taken componentwise:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \epsilon(g)_* \mathcal{F}_{n+1} & \longrightarrow & \epsilon(g)_* \mathcal{F}_n & \longrightarrow & \epsilon(g)_* \mathcal{F}_{n-1} \longrightarrow \cdots \\ & & \downarrow \mu(g)_{n+1} & & \downarrow \mu(g)_n & & \downarrow \mu(g)_{n-1} \\ \cdots & \longrightarrow & \mathcal{F}_{n+1} & \longrightarrow & \mathcal{F}_n & \longrightarrow & \mathcal{F}_{n-1} \longrightarrow \cdots \end{array}$$

From there, sheaves of E -vector spaces and $\overline{\mathbb{Q}}_\ell$ -sheaves can also be endowed with profinite group actions. The notion of a continuous action also extends to these categories by demanding that $\mu(g)_n$ satisfy Definition 2.1.2 for all n .

Our goal now is to prove a version of Proposition 2.1.3 for $\overline{\mathbb{Q}}_\ell$ -local systems on a scheme X over a field K with a K -rational point. By Proposition 1.2.3, we can represent a rank 1 $\overline{\mathbb{Q}}_\ell$ -local system with an ℓ -adic character of the étale fundamental group of X . The claim is that the homotopy exact sequence recalled in Section 1.1.6 in fact presents us with Deligne's equivalence for these local systems on X .

Let $e: \text{Spec}(K) \rightarrow X$ be a K -rational point of X , and \bar{e} the geometric point

corresponding to a choice of embedding $K \hookrightarrow \overline{K}$. Then, as described in Section 1.1.6, the functorial map

$$\pi_1(e): \pi_1(\mathrm{Spec}(K), \overline{e}) = \mathrm{Gal}(\overline{K}/K) \rightarrow \pi_1(X, \overline{e})$$

necessarily splits the homotopy exact sequence

$$1 \rightarrow \pi_1(\overline{X}, \overline{e}) \rightarrow \pi_1(X, \overline{e}) \rightarrow \mathrm{Gal}(\overline{K}/K) \rightarrow 1,$$

which allows us to write $\pi_1(X, \overline{e}) = \pi_1(\overline{X}, \overline{e}) \rtimes \mathrm{Gal}(\overline{K}/K)$. Because our attention is restricted to rank 1 $\overline{\mathbb{Q}}_\ell$ -local systems and thus to ℓ -adic characters of $\pi_1(X, \overline{e})$, which must factor through the respective abelianizations of the groups involved, we can consider the exact sequence

$$1 \rightarrow \pi_1(\overline{X}, \overline{e})_{\mathrm{Gal}(\overline{K}/K)} \rightarrow \pi_1(\overline{X}, \overline{e})_{\mathrm{Gal}(\overline{K}/K)} \times \mathrm{Gal}(K^{\mathrm{ab}}/K) \rightarrow \mathrm{Gal}(K^{\mathrm{ab}}/K) \rightarrow 1.$$

associated to the abelianization of $\pi_1(\overline{X}, \overline{e}) \rtimes \mathrm{Gal}(\overline{K}/K)$. This information alone tells us something very similar to Proposition 2.1.3—that every rank 1 ℓ -adic local system on X is equivalent to a rank 1 local system on \overline{X} together with a Galois action on $\overline{\mathbb{Q}}_\ell$, which is isomorphic to the stalks of the local system on \overline{X} . It is also isomorphic to the stalks of the local system on X as well, and what makes the claim less than immediate is the fact that the action indicated is in fact on the stalks of the local system there, not on \overline{X} . To prove the claim, we need to show that this Galois action can be translated to one as in Definition 2.1.1.

Let $\rho: \pi_1(X, \overline{x}) \rightarrow \overline{\mathbb{Q}}_\ell^\times$ be a character and \mathcal{F} a $\overline{\mathbb{Q}}_\ell$ -local system equivalent to

ρ under the correspondence of Proposition 1.2.3. We may assume that \mathcal{F} is itself represented by an E -local system $\mathcal{F} \otimes E$, where \mathcal{F} is a smooth π_E -adic sheaf and E is a finite extension of K . Each component torsion sheaf \mathcal{F}_n of \mathcal{F} may be taken to be the sheaf of sections of some Galois cover of X by the equivalence of finite locally constant sheaves and étale covers.

It is sufficient to prove the claim in the case of smooth π -adic sheaves to establish it for E -local systems since the latter category is just a quotient of the former and because the Galois action on an E -local system is defined through the π -adic sheaf giving rise to it. Likewise, the claim follows for $\overline{\mathbb{Q}}_\ell$ -local systems from the case of E -local systems.

We can make one further reduction and prove the claim for each constituent sheaf \mathcal{F}_n , since the fundamental group action is defined for each individually, arising from the action of the covering group on the Galois cover of X corresponding to \mathcal{F}_n under Proposition 1.1.2. We are left with the following statement.

Lemma 2.2.1. *Let \mathcal{F}_n be a finite locally constant sheaf on a scheme X , with a K -rational point e , equivalent to a character*

$$\rho_n: \pi_1(X, \bar{e}) = \pi_1(\overline{X}, \bar{e})_{\text{Gal}(\overline{E}/E)} \times \text{Gal}(E^{ab}/E) \rightarrow \overline{\mathbb{Q}}_\ell^\times.$$

Then, the natural Galois action on $\overline{\mathcal{F}}_n$ (in the sense of Definition 2.1.1) arising from base change to \overline{X} is equal to that defined by the action of $\pi_1(X, \bar{e})$ on \mathcal{F}_n and the inclusion

$$\text{Gal}(\overline{E}/E) \xrightarrow{\pi_1(e)} \pi_1(\overline{X}, \bar{e}) \rtimes \text{Gal}(\overline{E}/E).$$

Proof. We begin by calling into service the equivalence of Proposition 1.1.2, which

furnishes us with an étale cover Y_n whose sheaf of sections is the finite locally constant sheaf \mathcal{F}_n .

First, recall the action of the Galois group induced by the action of the fundamental group. The splitting of the fundamental group

$$\pi_1(X, \bar{e}) = \pi_1(\bar{X}, \bar{e}) \rtimes \text{Gal}(\bar{E}/E)$$

comes from the functorial map

$$\pi_1(e): \text{Gal}(\bar{E}/E) \longrightarrow \pi_1(X, \bar{e})$$

where e is the K -rational point of X . Let E_n be the finite Galois extension of E defined by the Cartesian square

$$\begin{array}{ccc} \text{Spec}(E_n) & \xrightarrow{e_n} & Y_n \\ \downarrow & & \downarrow \\ \text{Spec}(E) & \xrightarrow{e} & X \end{array}$$

Given $\gamma \in \text{Gal}(\bar{E}/E)$, $\pi_1(e)$ maps γ to the unique covering transformation φ_n of Y_n such that $e_n \circ \text{Spec}(\gamma) = \varphi_n \circ e_n$. Let U be étale over X . Then, φ_n induces a transformation of the cover $U \times_X Y_n \rightarrow U$ that we will denote φ_n^U . If $\sigma: U \rightarrow U \times_X Y_n$ is a section of the cover, so is $\varphi_n^U \circ \sigma$. Therefore $\sigma \mapsto \varphi_n^U \circ \sigma$ defines a permutation of the sections of \mathcal{F}_n over U . The action of the Galois group coming from the fundamental group action is defined through this action:

$$\gamma \cdot \sigma := \varphi_n^U \circ \sigma = \varphi_n^U \circ \sigma. \quad (2.2.1.1)$$

Next we consider the natural Galois sheaf structure obtained by base change. As explained in the discussion preceding Proposition 2.1.3, the isomorphisms giving this action are inverse limits of isomorphisms obtained by base changing to finite Galois extensions of E . For the sake of proper comparison, we retain the finite Galois extension E_n from above. Again, let $\gamma \in \text{Gal}(\overline{E}/E)$ and abusively write γ for its restriction to E_n as well. Let U be étale over X and U_{E_n} its base change to E_n . Consider the following diagram, in which all squares are Cartesian.

$$\begin{array}{ccccc}
 U_{E_n} \times_X Y_n & \xrightarrow[\gamma_{U \cap Y_n}]{\sim} & U_{E_n} \times_X Y_n & \longrightarrow & Y_n \\
 \downarrow & & \downarrow & & \downarrow \\
 U_{E_n} & \xrightarrow[\gamma_U]{\sim} & U_{E_n} & \longrightarrow & X \\
 \downarrow & & \downarrow & & \downarrow \\
 \text{Spec}(E_n) & \xrightarrow[\text{Spec}(\gamma)]{\sim} & \text{Spec}(E_n) & \longrightarrow & \text{Spec}(E)
 \end{array}$$

Set $X_{E_n} = X \times_{\text{Spec}(E)} \text{Spec}(E_n)$ and let $(\mathcal{F}_n)_{X_{E_n}}$ be the inverse image of \mathcal{F}_n on X_{E_n} .

The restriction of the Galois sheaf isomorphism to this level

$$\mu(\gamma)(U): \gamma_* \mathcal{F}_n(U) \xrightarrow{\sim} \mathcal{F}_n(U)$$

is simply the restriction map $(\mathcal{F}_n)_{X_{E_n}}(\gamma_U)$ associated to γ_U . To complete the comparison with the other action, we need to explicitly describe the action of these isomorphisms on sections of the covering $U_{E_n} \times_X Y_n \rightarrow U_{E_n}$.

Therefore, consider

$$\begin{array}{ccc}
 U_{E_n} \times_X Y_n & \xrightarrow[\gamma_{U \cap Y_n}]{\sim} & U_{E_n} \times_X Y_n \\
 \downarrow & & \sigma \updownarrow \\
 U_{E_n} & \xrightarrow[\gamma_U]{\sim} & U_{E_n}.
 \end{array}$$

σ being a section. The image of σ , $(\mu(\gamma)(U))^{-1}(\sigma)$, is determined by the universal property of $U_{E_n} \times_X Y_n$ using the two maps

$$\begin{array}{ccc}
 U_{E_n} & \xrightarrow{\sigma \circ \gamma_U} & U_{E_n} \times_X Y_n \\
 U_{E_n} & \xrightarrow{\text{id}} & U_{E_n}.
 \end{array}$$

The resulting map is easily seen to be

$$\mu(\gamma)(U)^{-1}(\sigma) = \gamma_{U \cap Y_n}^{-1} \circ \sigma \circ \gamma_U.$$

Thus,

$$\mu(\gamma)(U)(\sigma) = \gamma_{U \cap Y_n} \circ \sigma \circ \gamma_U^{-1}.$$

The equivalence of the two actions is confirmed by the following composite diagram.

$$\begin{array}{ccccccc}
 U_{E_n} \times_X Y_n & \xrightarrow[\gamma_{U \cap Y_n}]{\sim} & U_{E_n} \times_X Y_n & \xrightarrow{\pi_{Y_n}} & Y_n & \xleftarrow{\varphi_n} & Y_n \\
 \pi_U \downarrow & & \sigma \updownarrow \pi_U & & \downarrow & \swarrow & \\
 U_{E_n} & \xrightarrow[\gamma_U]{\sim} & U_{E_n} & \xrightarrow{\quad} & U & \xrightarrow{\quad} & X \\
 \pi_{E_n} \downarrow & & \downarrow \pi_{E_n} & & \downarrow & \nearrow e & \\
 \text{Spec}(E_n) & \xrightarrow[\text{Spec}(\gamma)]{\sim} & \text{Spec}(E_n) & \longrightarrow & \text{Spec}(E) & &
 \end{array}$$

Of particular importance is the commutating square

$$\begin{array}{ccc} U_{E_n} \times_X Y_n & \xrightarrow{\pi_{Y_n}} & Y_n \\ \sigma \updownarrow & & \downarrow \varphi_n \\ U_{E_n} & \xrightarrow{e_n \circ \pi_{E_n}} & Y_n, \end{array}$$

which implies that

$$\pi_{Y_n} \circ \sigma = \varphi_n \circ e_n \circ \pi_{E_n}. \quad (2.2.1.2)$$

Once again, the Galois sheaf action is given by

$$\begin{aligned} \gamma \cdot \sigma &= \mu(\gamma)(U)(\sigma) \\ &= \gamma_{U \cap Y_n}^{-1} \circ \sigma \circ \gamma_U, \end{aligned} \quad (2.2.1.3)$$

which we can more precisely characterize by taking the projections to the two components of the fibre product:

$$\begin{aligned} \pi_{U_{E_n}} \circ (\gamma \cdot \sigma) &= \pi_{U_{E_n}} \circ \gamma_{U \cap Y_n}^{-1} \circ \sigma \circ \gamma_U && \text{(by 2.2.1.3)} \\ &= \gamma_U^{-1} \circ \pi_{U_{E_n}} \circ \sigma \circ \gamma_U^{-1} \\ &= \text{id}_{U_{E_n}}. \end{aligned}$$

$$\begin{aligned}
\pi_{Y_n} \circ (\gamma \cdot \sigma) &= \pi_{Y_n} \circ \gamma_{U \cap Y_n}^{-1} \circ \sigma \circ \gamma_U && \text{(by 2.2.1.3)} \\
&= \text{id}_{Y_n} \circ \pi_{Y_n} \circ \sigma \circ \gamma_U^{-1} && \text{(by def'n of } \gamma_{U \cap Y_n}) \\
&= \varphi_n \circ e_n \circ \pi_{E_n} \circ \gamma_U && \text{(by 2.2.1.2)} \\
&= \varphi_n \circ e_n \circ \text{Spec}(\gamma) \circ \pi_{E_n} \\
&= \varphi_n \circ \varphi_n \circ e_n \circ \pi_{E_n} && \text{(by def'n of } \varphi_n) \\
&= \varphi_n \circ \pi_{Y_n} \circ \sigma && \text{(by 2.2.1.2)} \\
&= \pi_{Y_n} \circ (\text{id}_{U_{E_n}} \times \varphi_n) \circ \sigma \\
&= \pi_{Y_n} \circ \varphi_n^U \circ \sigma \\
&= \pi_{Y_n} \circ (\gamma \cdot \sigma), && \text{(by 2.2.1.1)}
\end{aligned}$$

where the action in the final line refers to the action of the Galois group induced by the fundamental group action described above in Equation 2.2.1.1. The Galois sheaf action is therefore trivial on the contribution from U_{E_n} and is identical to the Galois action from the fundamental group on the contribution from Y_n . We have therefore proved that a finite sheaf on a scheme over a field E with an action of the absolute Galois group on its stalk at a chosen geometric point is equivalent to a Galois sheaf, *i.e.*, the data of a sheaf on a scheme over the algebraic closure of E together with a family of isomorphisms as in Definition 2.1.1. \square

Proposition 2.2.2. *Let X be a scheme over K and suppose X has a K -rational point x with corresponding geometric point \bar{x} . Let $\pi_1(X, \bar{x}) = \pi_1(\bar{X}, \bar{x})_{\text{Gal}(\bar{K}/K)} \times \text{Gal}(\bar{K}/K)$ be the splitting of $\pi_1(X, \bar{x})^{ab}$ coming from $\pi_1(\bar{x})$.*

1. Let \mathcal{F} be a $\overline{\mathbb{Q}}_\ell$ -local system equivalent to

$$\rho: \pi_1(X, \overline{x}) \longrightarrow \overline{\mathbb{Q}}_\ell^\times.$$

Then, the natural compatible, continuous action of $\mathrm{Gal}(\overline{K}/K)$ arising from base change is equal to the action of $\mathrm{Gal}(\overline{K}/K)$ induced by the action of $\pi_1(X, \overline{x})$ and the Galois character

$$\mathrm{Gal}(\overline{K}/K) \xrightarrow{\pi_1(x)} \pi_1(X, \overline{x}) \xrightarrow{\rho} \overline{\mathbb{Q}}_\ell^\times.$$

2. The category of $\overline{\mathbb{Q}}_\ell$ -local systems on \overline{X} equivalent to characters that factor through $\pi_1(\overline{X}, \overline{x})_{\mathrm{Gal}(\overline{K}/K)}$, equipped with a continuous action of $\mathrm{Gal}(\overline{K}/K)$, is equivalent to the category of $\overline{\mathbb{Q}}_\ell$ -local systems on X .

Proof. Follows from Lemma 2.2.1, together with the homotopy exact sequence and Proposition 2.1.3, applied term by term to the component sheaves of \mathcal{F} . \square

Chapter 3

Some Local Systems on the Unipotent Cone in $\mathrm{SL}(2)$

This chapter is devoted to introducing the pair of local systems on the nilpotent cone in $\mathrm{SL}(2)_K$ whose nearby cycles will be studied in the next chapter, and to which we will associate distributions in Chapter 5. In the final section, we also establish that the local systems we have defined are equivariant with respect to the conjugation action of $\mathrm{SL}(2)_K$ on itself.

Let \mathcal{O} be a Henselian discrete valuation ring with field of fractions K and residue field k of characteristic $p > 0$. Assume also that the characteristic of K is 0.

3.1 $\mathrm{SL}(2)_K$ and the Unipotent Cone

We begin by describing the setting for the work done in the remainder of the thesis, the subvariety of regular unipotent elements $\mathcal{U}_{0,K}$ of the linear algebraic group $\mathrm{SL}(2)_K$. Our first step is to recall these schemes and to determine $\pi_1(\mathcal{U}_{0,K}, \bar{u})$, which will allow us to define the local systems.

3.1.1 The Schemes

In Section 3.2, we will define objects on the linear algebraic group

$$\mathrm{SL}(2) := \mathrm{Spec} (\mathbb{Z}[a, b, c, d]/(ad - bc - 1)),$$

a closed subgroup of $\mathrm{GL}(2) = \mathrm{Spec} (\mathbb{Z}[a, b, c, d]_{(ad-bc)})^1$. To abbreviate notation, we define

$$\mathbb{Z}[\mathrm{SL}(2)] := \mathbb{Z}[a, b, c, d]/(ad - bc - 1).$$

The group structure of $\mathrm{SL}(2)$ is defined by the following comultiplication, antipode, and identity maps.

1. Comultiplication:

$$m: \mathbb{Z}[\mathrm{SL}(2)] \longrightarrow \mathbb{Z}[\mathrm{SL}(2)] \otimes_{\mathbb{Z}} \mathbb{Z}[\mathrm{SL}(2)]$$

$$a \longmapsto a \otimes a + b \otimes c$$

$$b \longmapsto a \otimes b + b \otimes d$$

$$c \longmapsto a \otimes c + c \otimes d$$

$$d \longmapsto b \otimes c + d \otimes d$$

¹The notation here, which appears again several times below and on first blush might appear to mean localization at a prime ideal, actually means localization of the ring $K[a, b, c, d]$ at the multiplicative set composed of powers of $ad - bc$. Indeed, the ideal $(ad - bc)$ is not even prime.

2. Antipode:

$$\iota: \mathbb{Z}[\mathrm{SL}(2)] \longrightarrow \mathbb{Z}[\mathrm{SL}(2)]$$

$$a \longmapsto d$$

$$b \longmapsto -b$$

$$c \longmapsto -c$$

$$d \longmapsto a$$

3. Identity:

$$\mathbb{Z}[\mathrm{SL}(2)] \longrightarrow \mathbb{Z}$$

$$a \longmapsto 1$$

$$b \longmapsto 0$$

$$c \longmapsto 0$$

$$d \longmapsto 1$$

The familiar matrix group laws are recognizable in these maps and we will write the operations in their more familiar form (*i.e.*, on points) when convenient. From now on, write $G = \mathrm{SL}(2)_K := \mathrm{SL}(2) \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(K)$. For the coordinate ring of G , write $K[\mathrm{SL}(2)]$

The *unipotent* elements $g \in \mathrm{SL}(2)$ are those satisfying

$$(g - I)^n = 0$$

for some n . As a vanishing condition, this defines a closed subvariety of $\mathrm{SL}(2)$ for which we can find an alternate condition by simply considering the characteristic polynomial of such an element,

$$\begin{aligned}\chi_g(x) &= x^2 - \mathrm{tr}(g)x + \det(g) \\ &= x^2 - \mathrm{tr}(g)x + 1.\end{aligned}$$

The minimal polynomial $\min_g(x)$ of g divides this, and when it is a product of distinct linear factors, g is diagonalizable and thus not unipotent (the identity being the obvious exception). The only possibilities remaining are $\min_g(x) = (x + 1)^2$ or $\min_g(x) = (x - 1)^2$, the latter of which clearly corresponds to unipotent g . Therefore, the unipotent elements can alternately be characterized as elements with trace 2, and we can define the unipotent subvariety as the affine scheme

$$\mathcal{U} := \mathrm{Spec}(\mathbb{Z}[a, b, c, d]/(ad - bc - 1, a + d - 2)).$$

Once again, we will work with $\mathcal{U}_K = \mathcal{U} \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(K)$ and will simplify notation by setting

$$K[\mathcal{U}] := K[a, b, c, d]/(ad - bc - 1, a + d - 2).$$

The identity matrix is a singular point of this variety, and we will need to work on the smooth subvariety of regular unipotent elements obtained by removing this one singular (non-regular) point. The resulting scheme is no longer affine, but can be constructed by gluing together two schemes². The first is the open subscheme of \mathcal{U}_K

²For details on gluing schemes, see Hartshorne [16].

defined by the condition $b \neq 0$,

$$\mathcal{U}_{b \neq 0} := \operatorname{Spec}(\mathbb{Z}[\mathcal{U}]_b),$$

and the second by $c \neq 0$,

$$\mathcal{U}_{c \neq 0} := \operatorname{Spec}(\mathbb{Z}[\mathcal{U}]_c).$$

Both of these schemes contain the open subscheme where $b \neq 0 \neq c$,

$$\mathcal{U}_{b \neq 0, c \neq 0} := \operatorname{Spec}(\mathbb{Z}[\mathcal{U}]_{bc}).^3$$

The variety of regular (*i.e.*, nonsingular) unipotent elements \mathcal{U}_0 is obtained by gluing the schemes $\mathcal{U}_{b \neq 0}$ and $\mathcal{U}_{c \neq 0}$ along the open subscheme $\mathcal{U}_{b \neq 0, c \neq 0}$ by the identity isomorphism. Throughout the chapter, we will work with $\mathcal{U}_{0,K} := \mathcal{U}_0 \times \operatorname{Spec}(K)$.

3.1.2 Fundamental Groups

In the next section, we will define two $\overline{\mathbb{Q}}_\ell$ -local systems on $\mathcal{U}_{0,K}$ and extend these local systems to all of $G = \operatorname{SL}(2)_K$. By Proposition 1.2.3 it is possible to define such objects simply by specifying a continuous character of the étale fundamental group $\pi_1(\mathcal{U}_{0,K}, \overline{u})$. Therefore we need to determine this group.

³Once again, the rings in these definitions are localizations at the *elements* in the subscript, not at prime ideals

First, fix a geometric point \bar{u} of $\mathcal{U}_{0,K}$: Let

$$\begin{aligned} \bar{u}: K[\mathcal{U}]_{(a-1)}, K[\mathcal{U}]_{(d-1)} &\longrightarrow \bar{K} \\ a &\longmapsto 1 \\ b &\longmapsto 1 \\ c &\longmapsto 0 \\ d &\longmapsto 1; \end{aligned}$$

i.e.,

$$\bar{u} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

This map actually factors through the inclusion $K \hookrightarrow \bar{K}$, giving a K -rational point u , which splits the homotopy exact sequence from Section 1.1.6 applied to $\mathcal{U}_{0,K}$:

$$1 \rightarrow \pi_1(\bar{\mathcal{U}}_0, \bar{u}) \rightarrow \pi_1(\mathcal{U}_{0,K}, \bar{u}) \rightarrow \text{Gal}(\bar{K}/K) \rightarrow 1,$$

where $\bar{\mathcal{U}}_0 := \mathcal{U}_{0,K} \times_{\text{Spec}(K)} \text{Spec}(\bar{K})$. Thus, we only need to determine the fundamental group of $\bar{\mathcal{U}}_0$ to calculate $\pi_1(\mathcal{U}_{0,K}, \bar{u})$.

Lemma 3.1.1. $\pi_1(\bar{\mathcal{U}}_0, \bar{u}) \cong \mu_2$.

Proof. For this we deploy the comparison results Proposition 1.1.7 and Proposition 1.1.8, which tell us that $\pi_1(\bar{\mathcal{U}}_0, \bar{u}) \cong \widehat{\pi_1^{\text{top}}(\mathcal{U}_{0,\mathbb{C}})}$. As a complex manifold $\mathcal{U}_{0,\mathbb{C}}$

has a universal cover, the punctured affine plane $\mathbb{A}_{0,\mathbb{C}}^2$:

$$\begin{aligned} \mathbb{A}_{0,\mathbb{C}}^2 &\longrightarrow \mathcal{U}_{0,\mathbb{C}} \\ (x, y) &\longmapsto \begin{pmatrix} 1 - xy & x^2 \\ -y^2 & 1 + xy \end{pmatrix}. \end{aligned}$$

That $\mathbb{A}_{0,\mathbb{C}}^2$ is simply connected follows from the fact that it is homotopy equivalent to $S_{\mathbb{C}}^1 \cong S_{\mathbb{R}}^2$, which can be shown to be simply connected by invoking the Seifert-Van Kampen theorem⁴. Hence,

$$\pi_1^{\text{top}}(\mathcal{U}_{0,\mathbb{C}}) \cong \text{Aut}_{\mathcal{U}_{0,\mathbb{C}}}(\mathbb{A}_{0,\mathbb{C}}^2),$$

and this latter group consists of one nontrivial automorphism that maps x to $-x$ and y to $-y$. Thus, by Propositions 1.1.7 and 1.1.8,

$$\pi_1(\overline{\mathcal{U}}_0, \overline{u}) \cong \widehat{\pi_1^{\text{top}}(\mathcal{U}_{0,\mathbb{C}})} = \widehat{\text{Aut}_{\mathcal{U}_{0,\mathbb{C}}}(\mathbb{A}_{0,\mathbb{C}}^2)} = \widehat{\mu}_2 = \mu_2.$$

□

Lemma 3.1.2. $\pi_1(\mathcal{U}_{0,K}, \overline{u}) \cong \mu_2 \times \text{Gal}(\overline{K}/K)$.

Proof. The K -rational point u splits the homotopy exact sequence, which, by Lemma 3.1.1, yields

$$\pi_1(\mathcal{U}_{0,K}, \overline{u}) \cong \mu_2 \rtimes \text{Gal}(\overline{K}/K).$$

To see that the conjugation action of $\text{Gal}(\overline{K}/K)$ is trivial, consider an arbitrary con-

⁴See, for example, Munkres [31, Theorem 70.1].

nected Galois cover Y of $\mathcal{U}_{0,K}$. The automorphism group of Y over $\mathcal{U}_{0,K}$ contains at most one nontrivial “geometric” automorphism: the image of -1 under the functorial map from $\pi_1(\overline{\mathcal{U}}_0, \overline{u})$. As is shown in the proof of Lemma 3.2.3 below, this automorphism multiplies coordinates by -1 . The image of $\gamma \in \text{Gal}(\overline{K}/K)$ under $\pi_1(\overline{u})$ in the automorphism group of Y will either be the automorphism of Y induced by γ (which may be trivial), or that automorphism composed with the geometric automorphism. Conjugation of the geometric morphism by either of these will have no effect, since the geometric morphism is, like -1 , of order 2, and the natural induced Galois morphisms of Y commute with the geometric automorphism. Thus, the semidirect product above is in fact a direct product. \square

3.2 Local Systems on $\mathcal{U}_{0,K}$

With the precise form of $\pi_1(\mathcal{U}_{0,K}, \overline{u})$ now in hand, the next step is to define the local systems we will study on $\mathcal{U}_{0,K}$. Let ϖ be a uniformizer of the ring of integers of K , \mathcal{O} , and define the morphism of K -varieties $m(\varpi): G \rightarrow G$ by

$$m(\varpi): K[\text{SL}(2)] \longrightarrow K[\text{SL}(2)]$$

$$a \longmapsto a$$

$$b \longmapsto \frac{b}{\varpi}$$

$$c \longmapsto \varpi c$$

$$d \longmapsto d.$$

On points, this is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & \frac{b}{\varpi} \\ \varpi c & d \end{pmatrix}$$

Definition 3.2.1. Let

$$\rho_{\circ\mathcal{E}}: \pi_1(\mathcal{U}_{0,K}, \bar{u}) \longrightarrow \overline{\mathbb{Q}}_{\ell}$$

be the product of the nontrivial character of μ_2 and the trivial character of $\text{Gal}(\overline{K}/K)$, with respect to the product decomposition $\pi_1(\mathcal{U}_{0,K}, \bar{u}) \cong \mu_2 \times \text{Gal}(\overline{K}/K)$ defined by the choice of K -rational point

$$u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let ${}^{\circ}\mathcal{E}$ be the local system on $\mathcal{U}_{0,K}$ corresponding to $\rho_{\circ\mathcal{E}}$ under the equivalence of Proposition 1.2.3. In addition, let

$${}^{\circ}\mathcal{E}' := m(\varpi)^*({}^{\circ}\mathcal{E}).$$

Next, we study some alternate ways of defining ${}^{\circ}\mathcal{E}$ and ${}^{\circ}\mathcal{E}'$ that will aid in the proof of Theorem 4.3.1. In particular, we are interested in defining the local systems through characters of the automorphism groups of specific Galois covers of $\mathcal{U}_{0,K}$.

These covers will be the punctured affine plane $\mathbb{A}_{0,K}^2$. Like $\mathcal{U}_{0,K}$, this is a quasiaffine scheme constructed by gluing the two copies of \mathbb{A}^2 , one with the x -axis deleted, and

second with the y -axis deleted,

$$\mathbb{A}_{y \neq 0}^2 := \operatorname{Spec}(\mathbb{Z}[x, y]_y)$$

and

$$\mathbb{A}_{x \neq 0}^2 := \operatorname{Spec}(\mathbb{Z}[x, y]_x)$$

along the common open subscheme

$$\mathbb{A}_{x \neq 0 \neq y}^2 := \operatorname{Spec}(\mathbb{Z}[x, y]_{xy})$$

with the identity isomorphism.

Remark 3.2.2. Just as \mathcal{U}_0 and \mathbb{A}_0^2 are contained in \mathcal{U} and \mathbb{A}^2 as open subschemes, both $\mathcal{U}_{0,K}$ and $\mathbb{A}_{0,K}^2$ are contained as open subvarieties in the affine varieties \mathcal{U}_K and \mathbb{A}_K^2 , and all of the Galois covers we will consider can be defined on these affine schemes and obtained by restriction to $\mathbb{A}_{0,K}^2$. This is the approach we will take because it is much cleaner than defining maps on components of the gluing data for $\mathcal{U}_{0,K}$ and $\mathbb{A}_{0,K}^2$. But it should be noted that the affine versions of these covers are not étale, as they all ramify at the identity in \mathcal{U}_K . Thus, the notions and methods of Chapter 1 are not accessible for these maps; restricting attention to the open subvarieties is a necessary step.

Lemma 3.2.3. *The character ρ_ε is obtained by inflating the nontrivial character of*

the degree 2 Galois cover given on points by

$$f: \mathbb{A}_{0,K}^2 \longrightarrow \mathcal{U}_{0,K}$$

$$(x, y) \longmapsto \begin{pmatrix} 1 - xy & x^2 \\ -y^2 & 1 + xy \end{pmatrix}$$

and defined on coordinate rings by

$$K[\mathcal{U}] \longrightarrow K[\mathbb{A}^2]$$

$$a \longmapsto 1 - xy$$

$$b \longmapsto x^2$$

$$c \longmapsto -y^2$$

$$d \longmapsto 1 + xy,$$

to all of $\pi_1(\mathcal{U}_{0,K}, \overline{u})$.

Proof. That f is really an étale morphism is confirmed using the Jacobean criterion for étaleness (see, for example, Freitag and Kiehl [12], Chapter I, Prop. 1.3).

The rest of the lemma follows from the fact that the functorial map

$$\pi_1(\text{pr}): \pi_1(\overline{\mathcal{U}}, \overline{u}) \longrightarrow \pi_1(\mathcal{U}_{0,K}, \overline{u})$$

coming from the canonical projection $\text{pr}: \overline{\mathcal{U}}_0 \rightarrow \mathcal{U}_{0,K}$ induces an isomorphism between $\pi_1(\overline{\mathcal{U}}_0, \overline{u})$ and $\text{Aut}_{\mathcal{U}_{0,K}}(f)$.

To see this, recall Equation 1.1.5.1 and the discussion surrounding it, which ex-

plained how the functorial map is defined. In particular, if pr_f is the canonical projection $\pi_1(\mathcal{U}_{0,K}, \bar{u}) \rightarrow \text{Aut}_{\mathcal{U}_{0,K}}(f)$, then the composition $\text{pr}_f \circ \pi_1(\text{pr})$ is defined by pulling back the diagram

$$\begin{array}{ccc} & \mathbb{A}_{0,K}^2 & \\ & \downarrow f & \\ \bar{\mathcal{U}}_0 & \xrightarrow{\text{pr}} & \mathcal{U}_{0,K} \end{array}$$

to give

$$\begin{array}{ccc} \mathbb{A}_{0,K}^2 & \xrightarrow{\text{pr}^{\mathbb{A}^2}} & \mathbb{A}_{0,K}^2 \\ \bar{f} \downarrow & & \downarrow f \\ \bar{\mathcal{U}}_0 & \xrightarrow{\text{pr}} & \mathcal{U}_{0,K}. \end{array}$$

Then, $\pi_1(\text{pr})$ is defined by the relation

$$(\text{pr}^{\mathbb{A}^2}) \circ \phi = \pi_1(\text{pr})(\phi) \circ (\text{pr}^{\mathbb{A}^2}).$$

The fibres of f at all points aside from the identity contain 2 points, and its automorphism group consists of one nontrivial map, denoted (-1) , that sends x to $-x$ and y to $-y$. Étale covers are stable under base change, hence \bar{f} is a cover as well. It is likewise degree 2 with a single nontrivial covering map defined the same way as (-1) , which we distinguish by denoting it by $\overline{(-1)}$. This cover is the universal cover of $\bar{\mathcal{U}}_0$ implied by the proof of Lemma 3.1.1—the re-use of the notation \bar{f} is no accident. Since $\text{pr}^{\mathbb{A}^2}$ is equal to the canonical projection $\mathbb{A}_{\bar{K}}^2 \rightarrow \mathbb{A}_K^2$, it is clear that the map

$$\pi_1(\bar{\mathcal{U}}_0, \bar{u}) \cong \text{Aut}_{\bar{\mathcal{U}}_0}(\bar{f}) \xrightarrow{\pi_1(\text{pr})} \text{Aut}_{\mathcal{U}_{0,K}}(f)$$

is an isomorphism.

To ensure that inflation of the nontrivial character of $\text{Aut}_{\mathcal{U}_{0,K}}(f)$ to $\pi_1(\mathcal{U}_{0,K}, \bar{u})$ gives the trivial character on $\text{Gal}(\bar{K}/K)$, the next step is to calculate the composition

$$\text{Gal}(\bar{K}/K) \xrightarrow{\pi_1(u)} \pi_1(\mathcal{U}_{0,K}, \bar{u}) \xrightarrow{\text{pr}_f} \text{Aut}_{\mathcal{U}_{0,K}}(f).$$

Let Y be any finite Galois cover of $\text{Spec}(K)$. Thus, $Y = \text{Spec}(L)$, where L is a finite Galois extension of K . The relevant Cartesian diagram to consider is

$$\begin{array}{ccc} \mathbb{A}_{0,L}^2 & \longleftarrow & \text{Spec}(L) \\ f_L \downarrow & & \downarrow \\ \mathcal{U}_{0,K} & \xleftarrow{\bar{u}} & \text{Spec}(K) \end{array}$$

where f_L is defined analogously to f . The covering group of f_L consists of the automorphism $(-1)_L$, defined analogously to $(-1) \in \text{Aut}_{\mathcal{U}_{0,K}}(f)$, and the automorphisms induced from elements of $\text{Gal}(L/K)$. Thus,

$$\text{Aut}_{\mathcal{U}_{0,K}}(f_L) \cong \text{Aut}_{\mathcal{U}_{0,K}}(f) \times \text{Gal}(L/K)^5$$

and

$$\text{Aut}_{\text{Spec}(K)}(\text{Spec}(L)) \cong \text{Gal}(L/K).$$

The top horizontal arrow in the Cartesian square is the morphism that factors \bar{u}

⁵Although it turns out to be true in this case, it is not necessarily true that this decomposition exactly reflects the splitting we have for the fundamental group. Compare, for example, what happens for the map f' in the proof of Lemma 3.2.4.

through $\text{Spec}(L)$; on coordinates, it is defined the same way as \bar{u} . This implies that

$$\text{Gal}(L/K) \xrightarrow{(\text{pr}_{f_L}) \circ \pi_1(u)} \text{Aut}_{\mathcal{U}_{0,K}}(f) \times \text{Gal}(L/K)$$

is an isomorphism onto the second term.

The final step is to calculate the maps

$$g: \text{Aut}_{\mathcal{U}_{0,K}}(f) \longrightarrow \text{Aut}_{\mathcal{U}_{0,K}}(f_L)$$

induced by the map of covers $\mathbb{A}_{0,L}^2 \rightarrow \mathbb{A}_{0,K}^2$ present for every finite Galois extension L of K . For this, we consider the commuting triangle

$$\begin{array}{ccc} \mathbb{A}_{0,K}^2 & \xrightarrow{\tau} & \mathbb{A}_{0,L}^2 \\ & \searrow f & \swarrow f_L \\ & \mathcal{U}_{0,K} & \end{array}$$

For $\phi \in \text{Aut}_{\mathcal{U}_{0,K}}(f)$, $g(\phi)$ is defined by the similar relation

$$\tau \circ \phi = g(\phi) \circ \tau,$$

from which we see that

$$g((-1)) = (-1)_L.$$

Together, these last two calculations show that the nontrivial character of $\text{Aut}_{\mathcal{U}_{0,K}}(f)$ inflates to $\rho \circ \varepsilon$ on $\pi_1(\mathcal{U}_{0,K}, \bar{u})$. \square

In what follows, we will need to make reference to a square root of the uniformizer

ϖ . Therefore, set $K' := K(\sqrt{\varpi}) = K[x]/(x^2 - \varpi)$.

Lemma 3.2.4. *The local system ${}^\circ\mathcal{E}'$ is equivalent to the ℓ -adic character of $\pi_1(\mathcal{U}_{0,K}, \bar{u})$ obtained by inflating the nontrivial character of the automorphism group of the degree 2 étale cover*

$$f': \mathbb{A}_{0,K}^2 \longrightarrow \mathcal{U}_{0,K}$$

$$(x, y) \longmapsto \begin{pmatrix} 1 - xy & \frac{x^2}{\varpi} \\ -\varpi y^2 & 1 + xy \end{pmatrix}.$$

This cover is given on coordinates by

$$K[\mathcal{U}] \longrightarrow K[\mathbb{A}^2]$$

$$a \longmapsto 1 - xy$$

$$b \longmapsto \frac{x^2}{\varpi}$$

$$c \longmapsto -\varpi y^2$$

$$d \longmapsto 1 + xy.$$

Proof. In view of Lemma 3.2.3, we can find a fundamental group character equivalent to ${}^\circ\mathcal{E}'$ by base changing the cover f by the automorphism $m(\varpi)$. The necessary automorphism in the commuting square

$$\begin{array}{ccc} \mathbb{A}_{0,K}^2 & \longrightarrow & \mathbb{A}_{0,K}^2 \\ f \downarrow & & \downarrow f \\ \mathcal{U}_{0,K} & \xrightarrow{m(\varpi)} & \mathcal{U}_{0,K} \end{array}$$

sends x to $\frac{x}{\sqrt{\varpi}}$ and y to $\sqrt{\varpi}y$ —it is not defined over K , nor over an étale extension of K . The Cartesian square to consider instead is

$$\begin{array}{ccc} \mathbb{A}_{0,K}^2 & \xlongequal{\quad} & \mathbb{A}_{0,K}^2 \\ f' \downarrow & & \downarrow f \\ \mathcal{U}_{0,K} & \xrightarrow{m(\varpi)} & \mathcal{U}_{0,K}, \end{array}$$

with f' defined as in the statement of the lemma. Like f , f' is a degree 2 étale cover, and

$$\mathrm{Aut}_{\mathcal{U}_{0,K}}(f') = \{\mathrm{id}, (-1)'\},$$

where $(-1)'$ is defined like (-1) . The map of automorphism groups is an isomorphism, and the character of $\mathrm{Aut}_{\mathcal{U}_{0,K}}(f')$ obtained by composition is the nontrivial one, as claimed. \square

Lemma 3.2.5. *Let $\pi_1(\mathcal{U}_{0,K}, \bar{u}) \cong \mu_2 \times \mathrm{Gal}(\overline{K}/K)$ be the decomposition induced by the choice of the K -rational point \bar{u} . Then, under the equivalence of Proposition 1.2.3, ${}^\circ\mathcal{E}'$ corresponds to the character*

$$\rho \circ \mathcal{E}' : \mu_2 \times \mathrm{Gal}(\overline{K}/K) \longrightarrow \overline{\mathbb{Q}}_\ell$$

satisfying

$$\begin{aligned} \rho \circ \mathcal{E}'|_{\mu_2} &= \mathrm{sgn} \\ \rho \circ \mathcal{E}'|_{\mathrm{Gal}(\overline{K}/K)}(\gamma) &= \begin{cases} 1 & \gamma|_{K'} = \mathrm{id}_{K'} \\ -1 & \gamma|_{K'} = \mathrm{conj}. \end{cases} \end{aligned}$$

Proof. First, we calculate the composite map

$$\pi_1(\overline{\mathcal{U}}_0, \overline{u}) \xrightarrow{\pi_1(\text{pr})} \pi_1(\mathcal{U}_{0,K}, \overline{u}) \xrightarrow{\text{pr}_{f'}} \text{Aut}_{\mathcal{U}_{0,K}}(f')$$

by looking at the Cartesian square

$$\begin{array}{ccc} \mathbb{A}_{0,K}^2 & \longrightarrow & \mathbb{A}_{0,K}^2 \\ \overline{f} \downarrow & & \downarrow f' \\ \overline{\mathcal{U}}_0 & \xrightarrow{\text{pr}} & \mathcal{U}_{0,K}. \end{array}$$

The map between the covering spaces is defined by

$$\begin{aligned} x &\longmapsto \frac{x}{\sqrt{\varpi}} \\ y &\longmapsto \sqrt{\varpi}y. \end{aligned}$$

Therefore, the composite map is an isomorphism of groups. For the other half of the decomposition, we calculate

$$\text{Gal}(L/K) \xrightarrow{\pi_1(u)} \pi_1(\mathcal{U}_{0,K}, \overline{u}) \xrightarrow{\text{pr}_{f'}} \text{Aut}_{\mathcal{U}_{0,K}}(f').$$

As in the proof of Lemma 3.2.3, for any finite Galois extension L of K , we have the Cartesian square

$$\begin{array}{ccc} \mathbb{A}_{0,L}^2 & \longleftarrow & \text{Spec}(L) \\ f'_L \downarrow & & \downarrow \\ \mathcal{U}_{0,K} & \xleftarrow{u} & \text{Spec}(K). \end{array}$$

The map $\text{Spec}(L) \rightarrow \mathbb{A}_{0,L}^2$ arises from the ring homomorphism

$$\begin{aligned} L[\mathbb{A}^2] &\longrightarrow L \\ x &\longmapsto \sqrt{\varpi} \\ y &\longmapsto 0 \end{aligned}$$

(so in fact, L must be an extension of K containing $\sqrt{\varpi}$). Therefore, taking

$$\text{Aut}_{\mathcal{U}_{0,K}}(f'_L) \cong \mu_2 \times \text{Gal}(L/K),$$

we obtain

$$(\text{pr}_f \circ \pi_1(\overline{u}))(\gamma) = \begin{cases} \gamma, & \gamma|_{K'} = \text{id}_{K'} \\ (-1) \circ \gamma, & \gamma|_{K'} = \text{conj}. \end{cases}$$

From this, it follows that the character of $\pi_1(\mathcal{U}_{0,K}, \overline{u})$ obtained by inflating the non-trivial character of $\text{Aut}_{\mathcal{U}_{0,K}}(f')$ is $\rho_{\circ \mathcal{E}'}$. \square

For the purposes of the proof of Theorem 4.3.1, it will be useful to characterize ${}^\circ \mathcal{E}$ and ${}^\circ \mathcal{E}'$ in terms of two more Galois covers of $\mathcal{U}_{0,K}$. In both cases, the covering space is the punctured affine plane $\mathbb{A}_{0,K'}^2$ over the quadratic extension K' defined above. Once again this is a nonaffine scheme, but, in tune with Remark 3.2.2, we will work with it as an open subscheme of the affine scheme $\mathbb{A}_{K'}^2$.

Theorem 3.2.6. *The local systems ${}^\circ \mathcal{E}$ and ${}^\circ \mathcal{E}'$ are equivalent to the characters of $\pi_1(\mathcal{U}_{0,K}, \overline{u})$ induced by inflation from characters of the automorphism groups of Galois*

covers

$$f_{\varpi}, f'_{\varpi}: \mathbb{A}_{0,K'}^2 \longrightarrow \mathcal{U}_{0,K}$$

defined on coordinate rings by

$$K[\mathcal{U}] \longrightarrow K'[x, y]$$

$$f_{\varpi}: a \longmapsto 1 - xy$$

$$b \longmapsto x^2$$

$$c \longmapsto -y^2$$

$$d \longmapsto 1 + xy$$

and

$$f'_{\varpi}: a \longmapsto 1 - xy$$

$$b \longmapsto \frac{x^2}{\varpi}$$

$$c \longmapsto \varpi y^2$$

$$d \longmapsto 1 + xy$$

and given on points by

$$f_{\varpi}: (x, y) \longmapsto \begin{pmatrix} 1 - xy & x^2 \\ -y^2 & 1 + xy \end{pmatrix}$$

and

$$f'_{\varpi} : (x, y) \mapsto \begin{pmatrix} 1 - xy & \frac{x^2}{\varpi} \\ -\varpi y^2 & 1 + xy \end{pmatrix}$$

Proof. The precise characters will be given below. Both f_{ϖ} and f'_{ϖ} have automorphism groups of order 4, consisting of id, φ_1 , φ_2 , and φ_3 defined by

$$\begin{aligned} \varphi_1 : x &\mapsto -x \\ y &\mapsto -y \\ \varphi_2 : \sqrt{\varpi} &\mapsto -\sqrt{\varpi} \\ x &\mapsto x \\ y &\mapsto y \\ \varphi_3 : &\varphi_2 \circ \varphi_1; \end{aligned}$$

i.e., φ_1 multiplies coordinates by -1 and φ_2 acts on coefficients according to the non-trivial Galois automorphisms. To avoid confusion, we will denote the automorphisms of f'_{ϖ} by φ'_i , $i = 1, 2, 3$. To see how the characters corresponding to our local systems relate to characters of these automorphism groups, we need to explicitly find the four maps

$$\begin{array}{ccc} & & \text{Aut}_{\mathcal{U}_{0,K}}(f_{\varpi}) \\ & \nearrow \pi_{f_{\varpi}} & \\ \pi_1(\overline{\mathcal{U}}_0, \overline{u}) & \xrightarrow{\pi_1(\text{pr})} \pi_1(\mathcal{U}_{0,K}, \overline{u}) & \\ & \searrow \pi_{f'_{\varpi}} & \\ & & \text{Aut}_{\mathcal{U}_{0,K}}(f'_{\varpi}) \end{array}$$

and

$$\begin{array}{ccc}
 & & \text{Aut}_{\mathcal{U}_{0,K}}(f_{\varpi}) \\
 & \nearrow \pi_{f_{\varpi}} & \\
 \text{Gal}(\overline{K}/K) & \xrightarrow{\pi_1(u)} & \pi_1(\mathcal{U}_{0,K}, \overline{u}) \\
 & \searrow \pi_{f'_{\varpi}} & \\
 & & \text{Aut}_{\mathcal{U}_{0,K}}(f'_{\varpi}),
 \end{array}$$

where the leftmost maps are the functorial maps coming from, respectively, the projection $\overline{\mathcal{U}}_0 \rightarrow \mathcal{U}_{0,K}$ and the K -rational point $\overline{u}: \text{Spec}(K) \rightarrow \mathcal{U}_{0,K}$, and the maps $\pi_{f_{\varpi}}$ and $\pi_{f'_{\varpi}}$ are the canonical projections. The discussion around Equation 1.1.5.1 details how to find these compositions. For the first pair, we must consider the commuting diagram

$$\begin{array}{ccccccc}
 \mathbb{A}_{0,\overline{K}}^2 & \xrightarrow{\phi} & \mathbb{A}_{0,\overline{K}}^2 & \xrightarrow{\beta} & \mathbb{A}_{0,K'}^2 & \xrightarrow{\pi_1(\pi)(\phi)} & \mathbb{A}_{0,K'}^2 \\
 & \searrow \overline{f}_{\varpi}, \overline{f}'_{\varpi} & \downarrow & & \downarrow f_{\varpi}, f'_{\varpi} & \swarrow & \\
 & & \overline{\mathcal{U}}_{0,K} & \xrightarrow{\pi} & \mathcal{U}_{0,K} & &
 \end{array}$$

where the centre square is Cartesian, defined by the right and bottom sides. On coordinate rings, the map β is the natural inclusion $K'[x, y] \hookrightarrow \overline{K}[x, y]$ defined by a choice of embedding $K' \hookrightarrow \overline{K}$ (which is part of the choice of geometric point for the covers). The maps $\pi_1(\overline{\mathcal{U}}_0, \overline{u}) \rightarrow \text{Aut}_{\mathcal{U}_{0,K}}(f_{\varpi}), \text{Aut}_{\mathcal{U}_{0,K}}(f'_{\varpi})$ are then defined by the relation

$$\beta \circ \phi = \pi_1(\pi)(\phi) \circ \beta.$$

The cover $\mathbb{A}_{0,\overline{K}}^2 \rightarrow \overline{\mathcal{U}}_0$ along the left side of the Cartesian square is defined on coordinates in the same way as f_{ϖ} or f'_{ϖ} , depending on which lies on the right side of the square, and in either case has automorphism group of order 2, consisting of the

identity and nontrivial map (-1) , which sends x to $-x$ and y to $-y$. From this, we see that the two compositions are given by

$$\begin{aligned}
 \pi_1(\overline{\mathcal{U}}_0, \overline{u}) &\longrightarrow \text{Aut}_{\mathcal{U}_{0,K}}(f_{\varpi}) \\
 \text{id} &\longmapsto \text{id} \\
 (-1) &\longmapsto \varphi_1; \\
 \pi_1(\overline{\mathcal{U}}_0, \overline{u}) &\longrightarrow \text{Aut}_{\mathcal{U}_{0,K}}(f'_{\varpi}) \\
 \text{id} &\longmapsto \text{id} \\
 (-1) &\longmapsto \varphi'_1.
 \end{aligned}$$

For the two compositions coming from the Galois group, recall the choice of K -rational point $\overline{u}: \text{Spec}(K) \rightarrow \mathcal{U}_{0,K}$:

$$\begin{aligned}
 \overline{u}: K[\mathcal{U}_0] &\longrightarrow K \\
 a &\longmapsto 1 \\
 b &\longmapsto 1 \\
 c &\longmapsto 0 \\
 d &\longmapsto 1.
 \end{aligned}$$

The compositions of $\pi_1(\overline{u}): \text{Gal}(\overline{K}/K) \rightarrow \pi_1(\mathcal{U}_{0,K}, \overline{u})$ with the canonical projections to $\text{Aut}_{\mathcal{U}_{0,K}}(f_{\varpi})$ and $\text{Aut}_{\mathcal{U}_{0,K}}(f'_{\varpi})$ are defined by considering the similar commuting

diagram

$$\begin{array}{ccccccc}
 \mathrm{Spec}(K') & \xrightarrow{\phi} & \mathrm{Spec}(K') & \xrightarrow{\beta} & \mathbb{A}_{0,K'}^2 & \xrightarrow{\pi_1(\overline{w})(\phi)} & \mathbb{A}_{0,K'}^2 \\
 & \searrow & \downarrow & & \downarrow f_{\overline{w}}, f'_{\overline{w}} & \swarrow & \\
 & & \mathrm{Spec}(K) & \longrightarrow & \mathcal{U}_{0,K} & &
 \end{array}$$

where again the middle square is Cartesian and the morphism β is the morphism determined by the choice of geometric point for the cover $\mathrm{Spec}(K')$. With $f_{\overline{w}}$ along the right side, β is defined by $(x, y) \mapsto (1, 0)$; with $f'_{\overline{w}}$, β maps (x, y) to $(\sqrt{\overline{w}}, 0)$. $\mathrm{Spec}(K')$ has two covering automorphisms, the identity and a conjugation morphism defined by $\sqrt{\overline{w}} \mapsto -\sqrt{\overline{w}}$. In this case we find that the composite maps are

$$\begin{array}{ccc}
 \mathrm{Gal}(\overline{K}/K) & \longrightarrow & \mathrm{Aut}_{\mathcal{U}_{0,K}}(f_{\overline{w}}) \\
 \mathrm{id} & \longmapsto & \mathrm{id} \\
 \mathrm{conj} & \longmapsto & \varphi_2 \\
 \mathrm{Gal}(\overline{K}/K) & \longrightarrow & \mathrm{Aut}_{\mathcal{U}_{0,K}}(f'_{\overline{w}}) \\
 \mathrm{id} & \longmapsto & \mathrm{id} \\
 \mathrm{conj} & \longmapsto & \varphi'_3.
 \end{array}$$

This information together with the lemmata above imply that ${}^\circ\mathcal{E}$ and ${}^\circ\mathcal{E}'$ are equivalent to fundamental group characters induced by the following characters of the

automorphism groups of f_{ϖ} and f'_{ϖ} .

$${}^{\circ}\mathcal{E}: \mathrm{Aut}_{\mathcal{U}_{0,K}}(f_{\varpi}) \longrightarrow \overline{\mathbb{Q}}_{\ell}^{\times}$$

$$\mathrm{id} \longmapsto 1$$

$$\varphi_1 \longmapsto -1$$

$$\varphi_2 \longmapsto 1$$

$$\varphi_3 \longmapsto -1$$

$$\mathrm{Aut}_{\mathcal{U}_{0,K}}(f'_{\varpi}) \longrightarrow \overline{\mathbb{Q}}_{\ell}^{\times}$$

$$\mathrm{id} \longmapsto 1$$

$$\varphi_1 \longmapsto -1$$

$$\varphi_2 \longmapsto -1$$

$$\varphi_3 \longmapsto 1$$

$${}^{\circ}\mathcal{E}': \mathrm{Aut}_{\mathcal{U}_{0,K}}(f_{\varpi}) \longrightarrow \overline{\mathbb{Q}}_{\ell}^{\times}$$

$$\mathrm{id} \longmapsto 1$$

$$\varphi_1 \longmapsto -1$$

$$\varphi_2 \longmapsto -1$$

$$\varphi_3 \longmapsto 1$$

$$\begin{aligned}
\mathrm{Aut}_{\mathcal{U}_{0,K}}(f'_{\varpi}) &\longrightarrow \overline{\mathbb{Q}}_{\ell}^{\times} \\
\mathrm{id} &\longmapsto 1 \\
\varphi_1 &\longmapsto -1 \\
\varphi_2 &\longmapsto 1 \\
\varphi_3 &\longmapsto -1.
\end{aligned}$$

□

We end this section by depicting in the commutative diagram below the relationship between all of the Galois covers we have used.

$$\begin{array}{ccc}
\mathbb{A}_{0,K'}^2 & \xleftarrow{\tau} & \mathbb{A}_{0,K'}^2 \\
\downarrow & \swarrow f'_{\varpi} \quad \searrow f_{\varpi} & \downarrow \\
& \mathcal{U}_{0,K} & \\
\downarrow & \nwarrow f' \quad \nearrow f & \downarrow \\
\mathbb{A}_{0,K}^2 & & \mathbb{A}_{0,K}^2
\end{array}$$

where the arrow τ is an isomorphism of covers defined by

$$\begin{aligned}
x &\longmapsto \sqrt{\varpi}x \\
y &\longmapsto \frac{y}{\sqrt{\varpi}}
\end{aligned}$$

(therefore it is defined only over K' , not K). From the proof of Lemma 3.2.6 we can deduce that the map $\mathrm{Aut}_{\mathcal{U}_{0,K}}(f_{\varpi}) \rightarrow \mathrm{Aut}_{\mathcal{U}_{0,K}}(f'_{\varpi})$ induced by the isomorphism of

covers τ is given by

$$\begin{aligned} \mathrm{id} &\longmapsto \mathrm{id} \\ \varphi_1 &\longmapsto \varphi'_1 \\ \varphi_2 &\longmapsto \varphi'_3 \\ \varphi_3 &\longmapsto \varphi'_2. \end{aligned}$$

It is this ‘swapping’ of the latter two elements that is really the key to the result in Theorem 4.3.1.

3.3 Equivariance

In this section, we prove that the local systems ${}^\circ\mathcal{E}$ and ${}^\circ\mathcal{E}'$ are equivariant with respect to the conjugation action of $\mathrm{SL}(2)_K$ on $\mathcal{U}_{0,K}$. All notation established in the previous sections remains in effect, unless otherwise stated.

As the reader may discover by consulting Bernstein and Lunt’s book [5], the most general notion of equivariant sheaves is rather complicated. Here we use a simpler definition sufficient for the requirements of this thesis.

Definition 3.3.1. Let G be a linear algebraic group and X a scheme over a field K . If $\alpha: G \times X \rightarrow X$ is an action of G on X defined over K , and \mathcal{F} is a $\overline{\mathbb{Q}}_\ell$ -local system on X , \mathcal{F} is *G-equivariant* if there exists an isomorphism

$$\alpha^*\mathcal{F} \xrightarrow{\sim} p^*\mathcal{F},$$

where $p: G \times X \rightarrow X$ is the canonical projection.

Definition 3.3.2. The conjugation action $\text{conj}: \text{SL}(2)_K \times \mathcal{U}_K \longrightarrow \mathcal{U}_K$ of $\text{SL}(2)_K$ on \mathcal{U}_K is defined on coordinate rings by

$$\begin{aligned} \text{conj}: K[\mathcal{U}_0] &\longrightarrow K[\alpha, \beta, \gamma, \delta]/(\alpha\delta - \beta\gamma - 1) \otimes_K K[\mathcal{U}_0] \\ a &\longmapsto \alpha\delta \otimes a + \beta\delta \otimes c - \alpha\gamma \otimes b - \beta\gamma \otimes d \\ b &\longmapsto \alpha^2 \otimes b + \alpha\beta \otimes d - \alpha\beta \otimes a - \beta^2 \otimes c \\ c &\longmapsto \gamma\delta \otimes a + \delta^2 \otimes c - \gamma^2 \otimes b - \gamma\delta \otimes d \\ d &\longmapsto \alpha\gamma \otimes b + \alpha\delta \otimes d - \beta\gamma \otimes a - \beta\delta \otimes c \end{aligned}$$

Since the identity is a fixed point, this action restricts to one on $\mathcal{U}_{0,K}$. The projection map $p: \text{SL}(2)_K \times \mathcal{U}_{0,K} \rightarrow \mathcal{U}_{0,K}$ is defined on coordinate rings as

$$\begin{aligned} p: K[\mathcal{U}_0] &\longrightarrow K[\alpha, \beta, \gamma, \delta]/(\alpha\delta - \beta\gamma - 1) \otimes_K K[\mathcal{U}_0] \\ a &\longmapsto 1 \otimes a \\ b &\longmapsto 1 \otimes b \\ c &\longmapsto 1 \otimes c \\ d &\longmapsto 1 \otimes d. \end{aligned}$$

To prove equivariance, it will be necessary to relate the conjugation action to the following action of $\text{SL}(2)_K$ on \mathbb{A}_K^2 .

Definition 3.3.3. The action

$$\lambda: \mathrm{SL}(2)_K \times \mathbb{A}_K^2 \longrightarrow \mathbb{A}_K^2$$

$$\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, (x, y) \right) \longmapsto (\alpha x + \beta y, \gamma x + \delta y)$$

is given on coordinate rings by

$$\begin{aligned} \lambda: K[x, y] &\longrightarrow K[\alpha, \beta, \gamma, \delta]/(\alpha\delta - \beta\gamma - 1) \otimes_K K[x, y] \\ x &\longmapsto \alpha \otimes x + \beta \otimes y \\ y &\longmapsto \gamma \otimes x + \delta \otimes y. \end{aligned}$$

Lemma 3.3.4. *Both of the étale covers f and f' defined in Section 4.3 (see Lemmas 3.2.3 and 3.2.4) are morphisms of G -spaces wth respect to the conjugation action of Definition 3.3.2 and the linear action of Definition 3.3.3 on $\mathcal{U}_{0,K}$ and \mathbb{A}_K^2 , respectively. That is, the diagram*

$$\begin{array}{ccc} \mathrm{SL}(2)_K \times \mathbb{A}_K^2 & \xrightarrow{\lambda} & \mathbb{A}_K^2 \\ 1 \times f \downarrow & & \downarrow f \\ \mathrm{SL}(2)_K \times \mathcal{U}_{0,K} & \xrightarrow{c} & \mathcal{U}_{0,K}, \end{array}$$

as well as the same diagram with $1 \times f'$ along the left side and f' along the right, commutes.

Proof. By brute calculation. □

Proposition 3.3.5. *The local systems ${}^\circ\mathcal{E}$ and ${}^\circ\mathcal{E}'$ of Definition 3.2.1 are $\mathrm{SL}(2)_K$ -*

equivariant sheaves on $\mathcal{U}_{0,K}$, with respect to the conjugation action.

Proof. We give the proof for ${}^\circ\mathcal{E}$ only; the one for ${}^\circ\mathcal{E}'$ is totally analogous. Once again, the main tool for the proof is the equivalence between local systems and étale fundamental group representations. Therefore, we begin by fixing a geometric point $\bar{x}: \text{Spec}(\bar{K}) \rightarrow \text{SL}(2)_K \times \mathcal{U}_{0,K}$ of the product:

$$\begin{aligned} \bar{x}: K[\alpha, \beta, \gamma, \delta]/(\alpha\delta - \beta\gamma - 1) &\longrightarrow \bar{K} \\ (\alpha, \beta, \gamma, \delta, a, b, c, d) &\longmapsto (1, 0, 1, 0, 1, 1, 0, 1). \end{aligned}$$

Note that $c(\bar{x}) = \bar{u}$ and $p(\bar{x}) = \bar{u}$, where \bar{u} is the (K -rational) geometric point of $\mathcal{U}_{0,K}$ fixed throughout this chapter. Thus, we have two functorial maps between étale fundamental groups

$$\pi_1(\text{SL}(2)_K \times \mathcal{U}_{0,K}, \bar{x}) \xrightarrow{\pi_1(\text{conj})} \pi_1(\mathcal{U}_{0,K}, \bar{u})$$

and

$$\pi_1(\text{SL}(2)_K \times \mathcal{U}_{0,K}, \bar{x}) \xrightarrow{\pi_1(p)} \pi_1(\mathcal{U}_{0,K}, \bar{u}).$$

The inverse images $\text{conj}^*{}^\circ\mathcal{E}$ and $p^*{}^\circ\mathcal{E}$ are local systems, equivalent to the characters $\rho_{\circ\mathcal{E}} \circ \pi_1(\text{conj})$ and $\rho_{\circ\mathcal{E}} \circ \pi_1(p)$ of $\pi_1(\text{SL}(2)_K \times \mathcal{U}_{0,K}, \bar{x})$, respectively. From the definition of the functorial map and the alternate characterizations of $\rho_{\circ\mathcal{E}}$, it follows that these characters are defined entirely through characters of the automorphism groups of covers of $\text{SL}(2)_K \times \mathcal{U}_{0,K}$ we shall denote C and P . They are obtained by pullback

from the cover used to define ${}^\circ\mathcal{E}$, like so:

$$\begin{array}{ccc}
 C = \mathrm{SL}(2)_K \times \mathcal{U}_{0,K} \times_{\mathcal{U}_{0,K}} \mathbb{A}_K^2 & \longrightarrow & \mathbb{A}_K^2 \\
 \downarrow & & \downarrow f \\
 \mathrm{SL}(2)_K \times \mathcal{U}_{0,K} & \xrightarrow{c} & \mathcal{U}_{0,K}
 \end{array}$$

$$\begin{array}{ccc}
 P = \mathrm{SL}(2)_K \times \mathcal{U}_{0,K} \times_{\mathcal{U}_{0,K}} \mathbb{A}_K^2 & \longrightarrow & \mathbb{A}_K^2 \\
 \downarrow & & \downarrow f \\
 \mathrm{SL}(2)_K \times \mathcal{U}_{0,K} & \xrightarrow{p} & \mathcal{U}_{0,K}
 \end{array} .$$

The isomorphism

$$\mathrm{conj}^{*\circ}\mathcal{E} \cong p^{*\circ}\mathcal{E}$$

exists because the covers C and P are isomorphic, and thus have isomorphic auto-

morphism groups. The map $C \rightarrow P$ is defined on coordinates by

$$\begin{aligned}
 \alpha &\longrightarrow \alpha \\
 \beta &\longrightarrow \beta \\
 \gamma &\longrightarrow \gamma \\
 \delta &\longrightarrow \delta \\
 a &\longrightarrow a \\
 b &\longrightarrow b \\
 c &\longrightarrow c \\
 d &\longrightarrow d \\
 x &\longrightarrow \delta \otimes x - \beta \otimes y \\
 y &\longrightarrow \alpha \otimes y - \gamma \otimes x.
 \end{aligned}$$

This does indeed define a homomorphism since, in the coordinate ring for C , the first eight coordinates are related to the images of x and y under f by the conjugation map, so by Lemma 3.3.4, the images of x and y must be their images under the inverse group action on \mathbb{A}_K^2 . The inverse map $P \rightarrow C$ is defined as the identity on all coordinates except x and y , where

$$\begin{aligned}
 x &\longrightarrow \alpha \otimes x + \beta \otimes y \\
 y &\longrightarrow \gamma \otimes x + \delta \otimes y.
 \end{aligned}$$

Then the composition of these maps is

$$\begin{aligned}
 x &\longrightarrow \delta\alpha \otimes x - \beta\gamma \otimes x + \delta\beta \otimes y - \beta\delta \otimes y \\
 &= (\alpha\delta - \beta\gamma) \otimes x \\
 &= 1 \otimes x, \\
 y &\longrightarrow \alpha\delta \otimes x + \alpha\delta \otimes y - \alpha\delta \otimes x - \gamma\beta \otimes y \\
 &= (\alpha\delta - \gamma\beta) \otimes y \\
 &= 1 \otimes y
 \end{aligned}$$

and therefore the two maps are inverse isomorphisms, proving the claim. Thus, the two composite characters are the same, proving there is an isomorphism

$$\mathrm{conj}^{*\circ}\mathcal{E} \cong p^{*\circ}\mathcal{E}$$

as desired. □

Chapter 4

The Main Result

Finally we arrive at the main result of the thesis. It concerns the nearby cycles of the local systems ${}^\circ\mathcal{E}$ and ${}^\circ\mathcal{E}'$ defined in the previous chapter taken with respect to two different integral models for the regular unipotent subvariety $\mathcal{U}_{0,K}$. But first, we describe the nearby cycles functor itself.

Throughout Chapter 4, we retain the notation established in the previous chapter. In particular, K is a p -adic field of mixed characteristic and K' is the quadratic extension of K obtained by adjoining a square root of a uniformizer ϖ of K 's ring of integers, \mathcal{O}_K .

4.1 The Nearby Cycles Functor

Chapter 3 introduced a pair of local systems on the regular unipotent subvariety of $\mathrm{SL}(2)_K$ (K a local field of mixed characteristic) to which we will associate distributions in Chapter 5. The first step in that process makes use of the Grothendieck-Lefschetz trace formula, which only makes sense for local systems over the residue field k of K . Therefore, what's required is a mechanism that produces local systems on $\mathrm{SL}(2)_k$ from local systems on $\mathrm{SL}(2)_K$. The nearby cycles functor is just such a mechanism, one especially well-suited to this purpose because it also allows the local system on $\mathrm{SL}(2)_k$ to retain the action of $\mathrm{Gal}(\overline{K}/K)$ present on the original local system that

was studied in Chapter 2.

SGA 7(II) [1] Exposé XIII is the fundamental source for the material in this section. There, Deligne defines the nearby cycles functor for torsion sheaves on a scheme defined over a Henselian trait.

Let S be a Henselian trait (*i.e.*, the spectrum of a Henselian discrete valuation ring R) with closed point s and generic point η , so s is the spectrum of the residue field of R and η is the spectrum of the quotient field K of R . Choose a separable closure \overline{K} of K to define the geometric generic fibre $\overline{\eta}$. Let \overline{S} be the spectrum of the valuation ring of \overline{K} and \overline{s} the residue field of that ring. Let ℓ be relatively prime to the residue characteristic of R , p . Let X be a scheme over S with generic and special fibres X_η and X_s as in the following diagram (both squares are Cartesian).

$$\begin{array}{ccccc} X_\eta & \xhookrightarrow{j} & X & \xleftarrow{i} & X_s \\ \downarrow & & \downarrow & & \downarrow \\ \eta & \xhookrightarrow{\quad} & S & \xleftarrow{\quad} & s \end{array}$$

Finally, let \overline{X} , $X_{\overline{\eta}}$, and $X_{\overline{s}}$ be the respective base extensions of X , X_η , and X_s to \overline{S} , $\overline{\eta}$, and \overline{s} as defined by the Cartesian diagram below, which sits over the diagram above.

$$\begin{array}{ccccc} X_{\overline{\eta}} & \xhookrightarrow{\overline{j}} & \overline{X} & \xleftarrow{\overline{i}} & X_{\overline{s}} \\ \downarrow & & \downarrow & & \downarrow \\ \overline{\eta} & \xhookrightarrow{\quad} & \overline{S} & \xleftarrow{\quad} & \overline{s} \end{array}$$

In SGA, the nearby cycles functor is defined for torsion sheaves \mathcal{F} on X , and consists of two component sheaves, one taking the inverse image of \mathcal{F} on X_s as input and the other taking the inverse image of \mathcal{F} on X_η . This latter component is the one that we

will define and use as the nearby cycles functor. It carries torsion sheaves on X_η into a topos denoted $X_{\bar{s}} \times_s \eta$.

Definition 4.1.1. An element of the topos $X_s \times_s \eta$ is a sheaf $\mathcal{F}_{\bar{\eta}}$ on X_s , the subscript indicating that it is equipped with a continuous action of $\text{Gal}(\bar{\eta}/\eta)$ compatible with the action on $X_{\bar{s}}$ (via the continuous map $\text{Gal}(\bar{\eta}/\eta) \rightarrow \text{Gal}(\bar{s}/s)$).

Definition 4.1.2. Let \mathcal{F} be a torsion sheaf on X_η and $\mu_{\mathcal{F}}$ the natural continuous, compatible action (in the sense of Definitions 2.1.1 and 2.1.2) on the inverse image $\bar{\mathcal{F}}$ of \mathcal{F} on $X_{\bar{\eta}}$ (as explained in the paragraphs following Definition 2.1.2). The *nearby cycles functor* Ψ is defined by

$$\mathcal{F} \longmapsto \Psi(\mathcal{F}) := (\bar{i}^* \bar{j}_* \bar{\mathcal{F}}, \Psi(\mu_{\mathcal{F}})),$$

where $R\Psi(\mu_{\mathcal{F}})$ is the continuous, compatible action of $\text{Gal}(\bar{\eta}/\eta)$ on $\bar{i}^* \bar{j}_* \bar{\mathcal{F}}$ obtained from $\mu_{\mathcal{F}}$ by ‘transport of structure,’ a value phrase deserving some illumination.

To wit, for each $\gamma \in \text{Gal}(\bar{\eta}/\eta)$ with image $\bar{\gamma}$ under the canonical map $\text{Gal}(\bar{\eta}/\eta) \rightarrow \text{Gal}(\bar{s}/s)$, we will specify an isomorphism

$$\Psi(\mu_{\mathcal{F}})(\gamma): \bar{\gamma}_* \bar{i}^* \bar{j}_* \bar{\mathcal{F}}_{\bar{\eta}} \longrightarrow \bar{i}^* \bar{j}_* \bar{\mathcal{F}}_{\bar{\eta}}$$

derived from the given isomorphism

$$\mu_{\mathcal{F}}(\gamma): \gamma_* \mathcal{F}_{\bar{\eta}} \longrightarrow \mathcal{F}_{\bar{\eta}}.$$

This notation is somewhat deceptive because the isomorphism is not just the image

of $\mu_{\mathcal{F}}$ under the functor Ψ . That is, however, where things start:

$$\Psi(\mu(\gamma)) : \bar{i}^* \bar{j}_* \gamma_* \mathcal{F}_{\bar{\eta}} \longrightarrow \bar{i}^* \bar{j}_* \mathcal{F}_{\bar{\eta}}.$$

Then, we use the fact that

$$\bar{j}_* \gamma_* = (\bar{j} \gamma)_* = (\gamma \bar{j})_* = \gamma_* \bar{j}_*$$

because we have the commuting square

$$\begin{array}{ccc} \bar{\eta} & \xrightarrow{\bar{j}} & \bar{S} \\ \gamma \downarrow & & \downarrow \gamma \\ \bar{\eta} & \xrightarrow{\bar{j}} & \bar{S} \end{array}$$

where the γ along the right is the scheme morphism induced by the restriction of the Galois morphism γ to the ring R . So we can think of $R\Psi(\mu(\gamma))$ as a morphism

$$\bar{i}^* \gamma_* \bar{j}_* \mathcal{F}_{\bar{\eta}} \longrightarrow \bar{i}^* \bar{j}_* \mathcal{F}_{\bar{\eta}}.$$

The final step is provided by the proper base change theorem (Proposition 1.2.4), using the commuting square

$$\begin{array}{ccc} \bar{S} & \xleftarrow{\bar{i}} & \bar{s} \\ \gamma \downarrow & & \downarrow \bar{\gamma} \\ \bar{S} & \xleftarrow{\bar{i}} & \bar{s} \end{array}$$

where $\bar{\gamma}$ is the scheme morphism induced by the image of γ under the canonical homomorphism $\text{Gal}(\bar{\eta}/\eta) \rightarrow \text{Gal}(\bar{s}/s)$. The transported Galois isomorphism is the

composite

$$\overline{\gamma}_* \overline{i}^* \overline{j}_* \mathcal{F}_{\overline{\eta}} \cong \overline{i}^* \gamma_* \overline{j}_* \mathcal{F}_{\overline{\eta}} \longrightarrow \overline{i}^* \overline{j}_* \mathcal{F}_{\overline{\eta}},$$

the leftmost map coming from proper base change (see Theorem 1.2.4).

The functor Ψ has an obvious extension to the category of π -adic sheaves, and from there to $\overline{\mathbb{Q}}_\ell$ -local systems, using the definition of continuous, compatible actions extended to these categories in Chapter 2 Section 2.2.

Remark 4.1.3. As mentioned above, the original framework set by Deligne specifies the scheme X over the Henselian trait S at the outset, but the definition of Ψ given above only makes reference to X_η and X_s . Indeed, the work in this chapter begins with only the generic fibre X_η , leaving open the possibility of defining nearby cycles functors with respect to different integral models of X_η , *i.e.*, any scheme \underline{X} over S equipped with an isomorphism $\underline{X}_\eta \cong X_\eta$. Theorem 4.3.1 details what happens when we take the nearby cycles of the local systems ${}^\circ\mathcal{E}$ and ${}^\circ\mathcal{E}'$ from Chapter 3 with respect to two different integral models of $\mathcal{U}_{0,K}$. These integral models are defined in the next section.

4.2 Integral Models

Deligne defined nearby cycles for torsion sheaves on a scheme defined over a Henselian trait. Therefore, in the diagram

$$\begin{array}{ccccc} X_\eta & \xhookrightarrow{j} & X & \xleftarrow{i} & X_s \\ \downarrow & & \downarrow & & \downarrow \\ \eta & \xhookrightarrow{\quad} & S & \xleftarrow{\quad} & s \end{array}$$

the middle scheme X over the trait comes specified at the outset.

This thesis uses a ‘restriction’ of the nearby cycles functor, and begins only with a scheme over the field K , that is, with the generic fibre of the diagram. Thus, we are free to choose any integral model for the starting scheme and calculate nearby cycles with respect to that model. In the next section we will use two different models.

Definition 4.2.1. The *standard integral model* $\underline{\mathcal{U}}_0$ of $\mathcal{U}_{0,K}$ is obtained from the affine scheme

$$\underline{\mathcal{U}} := \mathcal{U}_{0,\mathcal{O}_K} = \operatorname{Spec}(\mathcal{O}_K[a, b, c, d]/(ad - bc - 1, a + d - 2))$$

by removing the identity. It can be built with a gluing construction analogous to that given for $\mathcal{U}_{0,K}$ (in § 3.1.1), with K replaced by \mathcal{O}_K throughout.

The *nonstandard integral model* $\underline{\mathcal{U}}'_0$ is obtained from the affine scheme

$$\underline{\mathcal{U}}' := \operatorname{Spec}\left(\mathcal{O}_K\left[a, \varpi b, \frac{c}{\varpi}, d\right]/(ad - bc - 1, a + d - 2)\right)$$

by likewise removing the identity. It too can be constructed by a gluing analogous to $\underline{\mathcal{U}}_0$ and \mathcal{U}_0 .

In accord with Remark 3.2.2, we will work with these models as if they were affine, conflating their coordinate rings with those of the affine schemes $\underline{\mathcal{U}}$ and $\underline{\mathcal{U}}'$, respectively. For compactness, we will denote these rings by $\mathcal{O}_K[\underline{\mathcal{U}}_0]$ and $\mathcal{O}_K[\underline{\mathcal{U}}'_0]$.

Here is how these models play into the main result: Recall that for a local system \mathcal{F} on $\mathcal{U}_{0,K}$, $\Psi\mathcal{F} = (\bar{i}^*\bar{j}_*\bar{\mathcal{F}}, \Psi(\mu))$, where $\bar{\mathcal{F}}$ is the inverse image of \mathcal{F} under the canonical projection $\bar{\mathcal{U}}_0 \rightarrow \mathcal{U}_{0,K}$, μ is the natural Galois action on $\bar{\mathcal{F}}$ induced by base change, and \bar{i} and \bar{j} are the inclusions of the special and generic fibres of a chosen integral model for $\mathcal{U}_{0,K}$. Since it is the inverse image of a local system, $\bar{\mathcal{F}}$ is a local system,

and so, in fact, is $\bar{i}^* \bar{j}_* \bar{\mathcal{F}}$, as will be shown in the proof of Theorem 4.3.1. If the character $\text{Gal}(\bar{K}/K)$ that defines the transported Galois action $\Psi(\mu)$ factors through the canonical map $\text{Gal}(\bar{K}/K) \rightarrow \text{Gal}(\bar{k}/k)$, giving a continuous action of $\text{Gal}(\bar{k}/k)$, then the extension of Deligne's equivalence proved in Chapter 2 shows that the nearby cycles is equivalent to a local system on $\mathcal{U}_{0,k}$. In this case, we say the nearby cycles *descends*.

Definition 4.2.2. When the nearby cycles of a local system, computed with respect to a particular integral model for $\mathcal{U}_{0,K}$, descends to a local system on $(\mathcal{U}_0)_k$ we will say that the integral model is *adapted* to the local system.

4.3 The Main Result

Theorem 4.3.1. *The integral model $\underline{\mathcal{U}}_0$ is adapted to the local system ${}^\circ\mathcal{E}$ but not to ${}^\circ\mathcal{E}'$, while $\underline{\mathcal{U}}'_0$ is adapted to ${}^\circ\mathcal{E}'$ but not to ${}^\circ\mathcal{E}$.*

Proof. The strategy of the proof comes from the following observation, justified at the end of the proof: that if the nearby cycles of a local system \mathcal{L} descends, the local system on $\mathcal{U}_{0,k}$ it descends to is $i^* j_* \mathcal{L}$, implying that this image sheaf is itself a local system. This is not the case generally, the obstruction being the finicky pushforward operation.

While inverse images of local systems are always local systems, and can be calculated at the level of fundamental group representations simply by composing the representation of the original sheaf with the functorial map between fundamental groups, the pushforward of a local system need not be a local system. Let $f: X \rightarrow Y$ be a morphism of schemes and \mathcal{F} a local system on X . For $f_* \mathcal{F}$ to be a local system,

the representation of $\pi_1(X, \overline{x})$ equivalent to \mathcal{F} must factor through the functorial map $\pi_1(f): \pi_1(X, \overline{x}) \rightarrow \pi_1(Y, \overline{y})$, as in

$$\begin{array}{ccc} \pi_1(X, \overline{x}) & \xrightarrow{\pi_1(f)} & \pi_1(Y, \overline{y}) \\ & \searrow \rho_{\mathcal{F}} \quad \swarrow \rho_{f_*\mathcal{F}} & \\ & \overline{\mathbb{Q}}_{\ell}^{\times} & \end{array}$$

in which case the factorizing character of $\pi_1(Y, \overline{y})$ is equivalent to $f_*\mathcal{F}$. Fortunately, the covers used to define ${}^{\circ}\mathcal{E}$ and ${}^{\circ}\mathcal{E}'$ arise from covers of $\underline{\mathcal{U}}_0$ and $\underline{\mathcal{U}}'_0$ via base change, which allows us to easily determine when the defining characters factor in this way and when they don't, from which the conclusions of the theorem derive.

In particular, the covers permitting these factorizations are by the affine plane over \mathcal{O}_K defined completely analogously to f and f' (from Lemmas 3.2.3 and 3.2.4) for the standard and nonstandard integral models, respectively. On (falsely affine) coordinate rings,

$$\begin{aligned} \underline{f}: \mathcal{O}_K[\underline{\mathcal{U}}] &\longrightarrow \mathcal{O}_K[x, y] \\ a &\longmapsto 1 - xy \\ b &\longmapsto x^2 \\ c &\longmapsto -y^2 \\ d &\longmapsto 1 + xy \end{aligned}$$

$$\begin{aligned}
\underline{f'}: \mathcal{O}_K[\underline{\mathcal{U}}'] &\longrightarrow \mathcal{O}_K[x, y] \\
a &\longmapsto 1 - xy \\
b &\longmapsto \frac{x^2}{\varpi} \\
c &\longmapsto -\varpi y^2 \\
d &\longmapsto 1 + xy.
\end{aligned}$$

Both of these covers have automorphism groups of order 2, with maps defined exactly as those for f and f' , which we will denote by $\underline{\text{id}}$ and $\underline{(-1)}$. The Cartesian square determined by the inclusion of the generic fibre into the standard integral model and the cover \underline{f} is

$$\begin{array}{ccc}
K[x, y] & \xleftarrow{\delta} & \mathcal{O}_K[x, y] \\
f \uparrow & & \uparrow \underline{f} \\
\mathcal{O}_K[\underline{\mathcal{U}}] & \longleftarrow & \mathcal{O}_K[\underline{\mathcal{U}}]
\end{array}$$

and the one determined by the inclusion into the nonstandard model and $\underline{f'}$ is

$$\begin{array}{ccc}
K[x, y] & \xleftarrow{\delta} & \mathcal{O}_K[x, y] \\
f' \uparrow & & \uparrow \underline{f'} \\
\mathcal{O}_K[\underline{\mathcal{U}}] & \longleftarrow & \mathcal{O}_K[\underline{\mathcal{U}}'].
\end{array}$$

In both cases, the homomorphism δ is simply the natural inclusion map. Therefore, the induced maps of automorphism groups are

$$\begin{aligned}
\text{Aut}_{\mathcal{U}_0, K}(f) &\longrightarrow \text{Aut}_{\underline{\mathcal{U}}_0}(\underline{f}) \\
\text{id} &\longmapsto \underline{\text{id}} \\
(-1) &\longmapsto \underline{(-1)}.
\end{aligned}$$

and

$$\begin{aligned} \mathrm{Aut}_{\mathcal{U}_{0,K}}(f') &\longrightarrow \mathrm{Aut}_{\underline{\mathcal{U}}'_0}(\underline{f}') \\ \mathrm{id} &\longmapsto \underline{\mathrm{id}} \\ (-1) &\longmapsto \underline{(-1)}, \end{aligned}$$

so that the nontrivial characters χ and χ' of $\mathrm{Aut}_{\mathcal{U}_{0,K}}(f)$ and $\mathrm{Aut}_{\mathcal{U}_{0,K}}(f')$ factor through the nontrivial characters $\underline{\chi}$ and $\underline{\chi}'$ of $\mathrm{Aut}_{\underline{\mathcal{U}}_0}(\underline{f})$ and $\mathrm{Aut}_{\underline{\mathcal{U}}'_0}(\underline{f}')$, respectively. If we define ${}^\circ\mathcal{E}$ and ${}^\circ\mathcal{E}'$ to be the local systems on $\underline{\mathcal{U}}_0$ and $\underline{\mathcal{U}}'_0$ defined by these characters, we can conclude that both $(j_{\underline{\mathcal{U}}_0})_* {}^\circ\mathcal{E}$ and $(j_{\underline{\mathcal{U}}'_0})_* {}^\circ\mathcal{E}'$ are local systems and specifically that

$$(j_{\underline{\mathcal{U}}_0})_* {}^\circ\mathcal{E} = {}^\circ\mathcal{E}$$

and

$$(j_{\underline{\mathcal{U}}'_0})_* {}^\circ\mathcal{E}' = {}^\circ\mathcal{E}'.$$

From here, there are no obstructions to the descent of the nearby cycles, since the inverse image of a local system always gives a local system. Composing $\underline{\chi}$ with the functorial map $\pi_1(i_{\underline{\mathcal{U}}_0}): \pi_1((\underline{\mathcal{U}}_0)_s, \bar{u}) \rightarrow \pi_1(\overline{\underline{\mathcal{U}}_0}, \bar{u})$ gives a character equivalent to a local system on $(\underline{\mathcal{U}}_0)_s = \mathcal{U}_{0,k}$ which we denote ${}^\circ\mathcal{E}_s$. Let $\overline{{}^\circ\mathcal{E}_s}$ be the base change of ${}^\circ\mathcal{E}_s$ to $\overline{(\underline{\mathcal{U}}_0)_s} := (\underline{\mathcal{U}}_0)_s \times_k \bar{k}$. Then,

$$\Psi_{\underline{\mathcal{U}}_0} {}^\circ\mathcal{E} = (\overline{{}^\circ\mathcal{E}_s}, \mu),$$

where μ is the natural action of $\mathrm{Gal}(\bar{K}/K)$ on $\overline{{}^\circ\mathcal{E}_s}$ arising from base change. And

composing $\underline{\chi}'$ with the functorial map $\pi_1(i_{\underline{\mathcal{U}}'_0}): \pi_1((\underline{\mathcal{U}}'_0)_s, \overline{u}) \rightarrow \pi_1(\underline{\mathcal{U}}'_0, \overline{u})$ gives a character equivalent to a local system on $(\underline{\mathcal{U}}'_0)_s$, denoted ${}^\circ\mathcal{E}'_s$. From this we conclude that

$$\Psi_{\underline{\mathcal{U}}'_0} {}^\circ\mathcal{E}' = (\overline{{}^\circ\mathcal{E}'_s}, \mu').$$

Under the equivalence of Proposition 2.1.3 extended to local systems in Chapter 2, both of these nearby cycles local systems descend to local systems on $(\underline{\mathcal{U}}_0)_s = (\underline{\mathcal{U}}'_0)_s$, which must be those local systems from which they were defined. This completes the proof of one half of the theorem.

The second half, regarding the nearby cycles of ${}^\circ\mathcal{E}$ and ${}^\circ\mathcal{E}'$ calculated with respect to the opposite integral model used previously, is proved by returning to the question of factoring the defining characters χ and χ' , this time factoring χ through $\pi_1(\underline{\mathcal{U}}'_0, \overline{u})$ and χ' through $\pi_1(\underline{\mathcal{U}}_0, \overline{u})$. In these cases, the claim is that no such factorizations exist, accounting for the failure of the nearby cycles to descend to local systems on $(\underline{\mathcal{U}}_0)_s = (\underline{\mathcal{U}}'_0)_s$.

The essential problem is that the only covers of the integral models from which the covers used to define ${}^\circ\mathcal{E}$ and ${}^\circ\mathcal{E}'$ can be obtained via base change are not étale, and therefore unusable for the purposes of factorizing χ and χ' . To see this we begin by noting that we could have approached the successful factorization above using the alternate covers f_ϖ and f'_ϖ through which ${}^\circ\mathcal{E}$ and ${}^\circ\mathcal{E}'$ can also be defined (by Lemma 3.2.6). Both of those covers arise by base change from covers of $\underline{\mathcal{U}}_0$ and $\underline{\mathcal{U}}'_0$, namely covers by $\mathbb{A}_{0, \mathcal{O}_{K'}}^2$ with covering maps defined in the same way as \underline{f} and \underline{f}' . However, neither of these covers is étale (both ramify, for example, at the points in $\underline{\mathcal{U}}_0$ and $\underline{\mathcal{U}}'_0$ corresponding to the prime ideal (p)), so the question of factorization could not be approached through those covers. But in both cases there existed intermediate

covers that did offer a means of factorizing the characters because those intermediate covers over $\mathcal{U}_{0,K}$ could also be used to define ${}^\circ\mathcal{E}$ and ${}^\circ\mathcal{E}'$ and the corresponding covers over the integral models were, in fact, étale. Similarly, there are Cartesian squares relating covers through which our local systems can be defined and covers of the standard and nonstandard integral models, namely

$$\begin{array}{ccc} \mathbb{A}_{0,K'}^2 & \longrightarrow & \mathbb{A}_{0,\mathcal{O}_{K'}}^2 \\ f_{\varpi} \downarrow & & \downarrow \underline{f}'_{\varpi} \\ \mathcal{U}_{0,K} & \longrightarrow & \underline{\mathcal{U}}'_0 \end{array}$$

and

$$\begin{array}{ccc} \mathbb{A}_{0,K'}^2 & \longrightarrow & \mathbb{A}_{0,\mathcal{O}_{K'}}^2 \\ f'_{\varpi} \downarrow & & \downarrow \underline{f}_{\varpi} \\ \mathcal{U}_{0,K} & \longrightarrow & \underline{\mathcal{U}}_0. \end{array}$$

But once again, this is not a usable setting to consider the question of factorization since the covers of the integral models are not étale. However, in this case there do not exist intermediate covers that define Cartesian subsquares of the ones above. The possible intermediate covers over $\mathcal{U}_{0,K}$ are f for the first diagram and f' for the second (because those are the only remaining covers that can be used to define the local systems we want), and the only possible covers over the integral models from which we could recover a Cartesian square are \underline{f}' for the first diagram and \underline{f} for the second. However, the only possible map between these pairs of covers that would complete such a square is defined only over the quadratic extension K'/K , not over K itself and therefore cannot be used (in particular, the necessary map would carry x to $\frac{x}{\sqrt{\varpi}}$ and y to $\sqrt{\varpi}y$). Hence, the character χ cannot factor through $\pi_1(\underline{\mathcal{U}}'_0, \overline{u})$ nor

can χ' factor through $\pi_1(\underline{\mathcal{U}}_0, \bar{u})$. We can infer, then, that the pushforward of ${}^\circ\mathcal{E}$ to $\underline{\mathcal{U}}'_0$ is not a local system, nor is the pushforward of ${}^\circ\mathcal{E}'$ to $\underline{\mathcal{U}}_0$ a local system.

This allows us to conclude that $\Psi_{\underline{\mathcal{U}}'_0} {}^\circ\mathcal{E}$ and $\Psi_{\underline{\mathcal{U}}_0} {}^\circ\mathcal{E}'$ do not descend to local systems on $(\underline{\mathcal{U}}_0)_s$. Justification for this conclusion comes from the following two facts: first, that the sheaf components of $\Psi_{\underline{\mathcal{U}}'_0} {}^\circ\mathcal{E}$ and $\Psi_{\underline{\mathcal{U}}_0} {}^\circ\mathcal{E}'$ are both local systems on $\mathcal{U}_{0,\bar{k}}$; and second, that if these local systems descended, the pushforwards that we attempted to calculate above would be local systems. The first fact is proved by first noting that the base changes of ${}^\circ\mathcal{E}$ and ${}^\circ\mathcal{E}'$ are local systems on $\bar{\mathcal{U}}_0$, defined by the compositions of χ and χ' with the functorial map

$$\pi_1(\bar{\mathcal{U}}_0, \bar{u}) \longrightarrow \pi_1(\mathcal{U}_{0,K}, \bar{u}).$$

From the calculations above, it can be seen that both of these compositions are the same, equal to the nontrivial character of $\pi_1(\bar{\mathcal{U}}_0, \bar{u})$. The first fact is proved after it is shown that the pushforward of this local system to the integral model

$$\bar{\mathcal{U}}_0 = \text{Spec}(\mathcal{O}_{K^{\text{nr}}}[a, b, c, d]/(ad - bc - 1, a + d - 2)),$$

where K^{nr} is the maximal unramified extension of K , is a local system. To do this, we construct a factorization of the nontrivial character of $\pi_1(\bar{\mathcal{U}}_0, \bar{u})$ through the automorphism group of the following étale cover of $\bar{\mathcal{U}}_0$. Let $\mathbb{A}_{\mathcal{O}_{\bar{K}}}^2 \rightarrow \bar{\mathcal{U}}_0$ be given on

coordinate rings by

$$\begin{aligned}
 \mathcal{O}_{\overline{K}}[a, b, c, d]/(ad - bc - 1, a + d - 2) &\longrightarrow \mathcal{O}_{\overline{K}}[x, y] \\
 a &\longmapsto 1 - xy \\
 b &\longmapsto x^2 \\
 c &\longmapsto -y^2 \\
 d &\longmapsto 1 + xy.
 \end{aligned}$$

This is a degree 2 Galois cover with automorphism group consisting of a nontrivial map that multiplies both coordinates by -1. The square

$$\begin{array}{ccc}
 \mathbb{A}_{\overline{K}}^2 & \longrightarrow & \mathbb{A}_{\mathcal{O}_{\overline{K}}}^2 \\
 \downarrow & & \downarrow \\
 \overline{\mathcal{U}}_0 & \xrightarrow{\overline{j}} & \underline{\mathcal{U}}_0
 \end{array}$$

is Cartesian, and the top map defines a map of automorphism groups that sends nontrivial map to nontrivial map, allowing the base changed character to factor. Therefore, $\overline{j}_* \overline{\circ \mathcal{E}} = \overline{j}_* \overline{\circ \mathcal{E}'}$ is a local system, and hence so is its inverse image on the special fibre.

If $\overline{j}_* \overline{\circ \mathcal{E}}$ descends to local system on $(\underline{\mathcal{U}}_0)_s$, its pushforwards to the integral model the nearby cycles were taken with respect to would be a local system, and equal to the pushforwards from the generic fibre we have attempted to calculate. This is because the trait $S = \text{Spec}(\mathcal{O}_K)$ both integral models are defined over is Henselian, and the étale site of a scheme defined over a Henselian trait is equivalent to the étale site of its special fibre (see Murre [32, 8.1.3]). This result renders the factorization requirement

for pushforwards of local systems to be local systems trivial for pushforwards from the special fibre (covers of the special fibre *always* arise via base change from the integral model and each such pair of covers will have isomorphic covering groups). This completes the proof. \square

Chapter 5

Distributions From The Local Systems

As an application of Theorem 4.3.1, the last piece of business in this thesis is to associate distributions to ${}^\circ\mathcal{E}$ and ${}^\circ\mathcal{E}'$ and establish some of the basic properties of those distributions. In particular, it turns out that they are admissible in the sense of Harish-Chandra [15] and are eigendistributions of the Fourier transform.

As in the previous chapter, set $G = \mathrm{SL}(2)_K$; in addition, we write G_k for $\mathrm{SL}(2)_k$.

5.1 Distributions from the Local Systems

By Theorem 4.3.1, the nearby cycles sheaves

$$\mathcal{E} := R\Psi_{\underline{\mathcal{U}}_0} {}^\circ\mathcal{E} \text{ and } \mathcal{E}' := R\Psi_{\underline{\mathcal{U}}'_0} {}^\circ\mathcal{E}'$$

on $\mathcal{U}_{0,\bar{k}}$ descend to local systems on $\mathcal{U}_{0,k}$. Both \mathcal{E} and \mathcal{E}' consist of two pieces of data, a local system on $\mathcal{U}_{0,\bar{k}}$ and a continuous action of $\mathrm{Gal}(\bar{k}/k)$. The proof of Theorem 4.3.1 showed that both components of \mathcal{E} and \mathcal{E}' are equal. Despite this we will continue refer to both because distinct objects will eventually be defined from these two nearby cycles sheaves. Denote the local system component of both sheaves by $\bar{\mathcal{E}}$. That \mathcal{E}

and \mathcal{E}' descend means there is an isomorphisms of sheaves

$$\varphi: \mathrm{Fr}^* \overline{\mathcal{E}} \xrightarrow{\sim} \overline{\mathcal{E}}$$

and

$$\varphi': \mathrm{Fr}^* \overline{\mathcal{E}'} \xrightarrow{\sim} \overline{\mathcal{E}'},$$

where Fr is the automorphism of $G_{\overline{k}}$ induced by the Frobenius in $\mathrm{Gal}(\overline{k}/k)$. Then, as described in SGA4 $\frac{1}{2}$ [10, *Rappels*], it is possible to define a characteristic function for each of \mathcal{E} and \mathcal{E}' .

Definition 5.1.1. The *trace of Frobenius* associated to a local system \mathcal{F} on $\mathcal{U}_{0,\overline{k}}$ with an isomorphism

$$\varphi_{\mathcal{F}}: \mathrm{Fr}^* \mathcal{F} \xrightarrow{\sim} \mathcal{F}$$

is given by

$$t_{\mathrm{Fr}}^{\mathcal{F}}(a) := \mathrm{Trace}((\varphi_{\mathcal{F}})_{\overline{a}}; \mathcal{F}_{\overline{a}}), \quad \forall a \in \mathcal{U}_{0,\overline{k}}(k).$$

To be precise, the value $t_{\mathrm{Fr}}^{\mathcal{F}}(a)$ is the trace of the operator on $\mathcal{F}_{\overline{a}}$ induced by the composition of morphisms

$$\mathcal{F}_{\overline{a}} = \mathcal{F}_{\mathrm{Fr}(\overline{a})} \xrightarrow{\sim} (\mathrm{Fr}^* \mathcal{F})_{\overline{a}} \xrightarrow{\varphi_{\overline{a}}} \mathcal{F}_{\overline{a}}, \quad (5.1.1.1)$$

the leftmost map being the canonical isomorphism and $\varphi_{\overline{a}}$ the isomorphism of stalks induced by φ .

Thus, both $t_{\mathrm{Fr}}^{\mathcal{E}}$ and $t_{\mathrm{Fr}}^{\mathcal{E}'}$ are functions on $\mathcal{U}_{0,k}(k)$. Extend these functions by zero to $G_k(k)$. From there, we can obtain functions on $G(K)$ using the parahoric subgroups

of $\mathrm{SL}(2)_K$ that sit above the integral models $\underline{\mathcal{U}}_0$ and $\underline{\mathcal{U}}'_0$. These are

$$\begin{aligned} X &:= \mathrm{Spec}(\mathcal{O}[a, b, c, d]/(ad - bc - 1)) \\ X' &:= \mathrm{Spec}\left(\mathcal{O}\left[a, \varpi b, \frac{c}{\varpi}, d\right]/(ad - bc - 1)\right), \end{aligned}$$

respectively. Inflate $t_{\mathrm{Fr}}^{\mathcal{E}}$ to $X(\mathcal{O}_K)$ and $t_{\mathrm{Fr}}^{\mathcal{E}'}$ to $X'(\mathcal{O}_K)$ by composing with the reduction map

$$X(\mathcal{O}_K) \text{ (resp. } X'(\mathcal{O}_K)) \longrightarrow G_k(k).$$

Finally, extend the results by zero to obtain functions

$$f_{\circ \mathcal{E}}, f_{\circ \mathcal{E}'}: G(K) \longrightarrow \overline{\mathbb{Q}}_{\ell}.$$

Each of these functions is supported by a compact subgroup—either $X(\mathcal{O}_K)$ or $X'(\mathcal{O}_K)$ —and the value of any point in the support is determined by its value under the reduction map. Therefore, both $f_{\circ \mathcal{E}}$ and $f_{\circ \mathcal{E}'}$ belong to the Hecke algebra of $G(K)$.

Definition 5.1.2. The *Hecke algebra* $\mathcal{C}(G(K))$ of $G(K)$ is the set of compactly supported and locally constant functions

$$f: G(K) \longrightarrow \overline{\mathbb{Q}}_{\ell}$$

endowed with the structure of a $\overline{\mathbb{Q}}_{\ell}$ -algebra with scalar multiplication, pointwise addition and convolution of functions.

Remark 5.1.3. It is worth noting here that although the trace of Frobenius functions $t_{\mathrm{Fr}}^{\mathcal{E}}$ and $t_{\mathrm{Fr}}^{\mathcal{E}'}$ are identical, they were inflated through different parahoric subgroups.

The resulting functions $f_{\circ\mathcal{E}}$ and $f_{\circ\mathcal{E}'}$ are distinct; therefore, the subscripts are no longer meaningless, keeping track of the parahoric subgroup the trace was inflated to. The switch to ${}^{\circ}\mathcal{E}$ and ${}^{\circ}\mathcal{E}'$ is meant to emphasize the connection to the original local systems on G that give rise to these functions.

Throughout the rest of this chapter we fix a Haar measure on $G(K)$.

Definition 5.1.4. Let $\mathcal{L} \in \{{}^{\circ}\mathcal{E}, {}^{\circ}\mathcal{E}'\}$. Then for all $f \in \mathcal{C}(G(K))$,

$$\Theta_{\mathcal{L}}(f) := \int_{G(K)} \int_{G(K)} f(y^{-1}xy) f_{\mathcal{L}}(x) dx dy.$$

5.2 Fundamental Properties of the Distributions

Now we prove that the integrals in Definition 5.1.4 do indeed converge, giving distributions, and that these distributions are *admissible* as defined by Harish-Chandra in [15, §14]. There, he goes on to note that all distributions obtained as characters of admissible representations (a process described in [14, §5]) as well as all linear combinations of such distributions are admissible. It turns out that $\Theta_{\circ\mathcal{E}}$ and $\Theta_{\circ\mathcal{E}'}$ are just such objects. In light of that, recalling the exact and somewhat technical definition of admissibility is something of an unnecessary distraction, so we will refrain. The curious reader can consult Harish-Chandra's article.

Proposition 5.2.1. *Let $\mathcal{L} \in \{\mathcal{E}, \mathcal{E}'\}$. The integral $\Theta_{\mathcal{L}}$ in Definition 5.1.4*

1. *converges for every $f \in \mathcal{C}(G(K))$, therefore defining a distribution;*
2. *is an admissible distribution, as defined by Harish-Chandra [15, §14].*

Proof. 1. We will prove this by relating $\Theta_{\mathcal{L}}$ to integrals that are known to converge.

To do this, it is necessary to first determine the trace of Frobenius functions

attached to \mathcal{E} and \mathcal{E}' in the previous section. Recall that

$$t_{\text{Fr}}^{\mathcal{E}}(a) = \text{Trace}((\varphi_{\overline{\mathcal{E}}})_a; \overline{\mathcal{E}}_a), \quad \forall a \in (\mathcal{U}_{0,k})(k).$$

What is required, then, is to find the action of the composition of isomorphisms in 5.1.1.1 on the stalk $\overline{\mathcal{E}}_a$.

We will do this calculation for the points

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad a' = \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix},$$

where ε is a quadratic nonresidue in \mathbb{F}_q^\times , since G -equivariance together with the fact that all points in $\mathcal{U}_{0,k}(k)$ are $G_k(k)$ -conjugate to one or the other of these points means that this is sufficient to know the trace function completely.

The local system $\overline{\mathcal{E}}$ is defined by a character χ_s of the étale cover

$$\mathbb{A}_{0,\overline{k}}^2 \xrightarrow{f_s} \mathcal{U}_{0,\overline{k}}.$$

This cover is given on coordinates by

$$a \longmapsto 1 - xy$$

$$b \longmapsto x^2$$

$$c \longmapsto -y^2$$

$$d \longmapsto 1 + xy.$$

Following Mars & Springer [27, §2], the stalk $\overline{\mathcal{E}}_{\overline{z}}$ at a K -rational point \overline{z} can be identified with

$$\{\phi: f_s^{-1}(z) \rightarrow \overline{\mathbb{Q}}_\ell \mid \gamma\phi = \chi_s(\gamma)\phi, \forall \gamma \in \pi_1(\mathcal{U}_{0,\overline{k}}, \overline{u})\};$$

i.e., the set of functions on the fibre of f_s above \overline{z} on which the fundamental group acts (via precomposition with the action of the fundamental group on elements of the fibre) according to the character that defines $\overline{\mathcal{E}}$. Similarly, $(\mathrm{Fr}^*\overline{\mathcal{E}})_{\overline{z}}$ is the set of functions on the image of $f_s^{-1}(z)$ under $\mathrm{Fr}: (\mathbb{A}_{\overline{k}}^2)_0(\overline{k}) \rightarrow (\mathbb{A}_{\overline{k}}^2)_0(\overline{k})$ satisfying the same restriction.

For the point \overline{a} , we have

$$f_s^{-1}(a) = \{(1, 0), (-1, 0)\}.$$

Fr acts trivially on these points, so the stalks $\overline{\mathcal{E}}_{\overline{a}}$ and $(\mathrm{Fr}^*\overline{\mathcal{E}})_{\overline{a}}$ are exactly the same, and the composition 5.1.1.1 consists of trivial isomorphisms. Therefore,

$$t_{\mathrm{Fr}}^{\mathcal{E}}(a) = 1.$$

On the other hand, $f_s^{-1}(a') = \{(\sqrt{\varepsilon}, 0), (-\sqrt{\varepsilon}, 0)\}$, on which Fr acts nontrivially, swapping the two elements. The stalks $\overline{\mathcal{E}}_{\overline{a'}}$ and $(\mathrm{Fr}^*\overline{\mathcal{E}})_{\overline{a'}}$ are thus the same set of functions, and the canonical isomorphism in the composition 5.1.1.1 is trivial, with trace 1. The second isomorphism, however, has trace -1, owing to the fact that the action of Frobenius on $f_s^{-1}(a')$ coincides with the action of the nontrivial element $\gamma \in \pi_1(\mathcal{U}_{0,\overline{k}}, \overline{u})$ (the method used in Chapter 4, Section 3.1.2

to determine $\pi_1(\overline{\mathcal{U}}_0, \overline{u})$ works on the special fibre as well, and yields an analogous result). If $\alpha' \in f_s^{-1}(a')$ and $\phi \in (\mathrm{Fr}^* \overline{\mathcal{E}})_{\overline{a}'}$,

$$\begin{aligned}
 (\varphi_{\overline{a}'}(\phi))(\alpha') &= \phi(\mathrm{Fr}^{-1}(\alpha')) \\
 &= \phi(\gamma \cdot \alpha') \\
 &= {}^\gamma \phi(\alpha') \\
 &= \chi_s(\gamma) \phi(\alpha') \\
 &= (-1) \phi(\alpha').
 \end{aligned}$$

Therefore,

$$t_{\mathrm{Fr}}^{\mathcal{E}}(a') = -1.$$

Since all elements of $\mathcal{U}_{0,k}$ are $\mathrm{SL}(2)_k$ -conjugate to either a or a' , and both \mathcal{E} and \mathcal{E}' are equivariant (this can be proved analogously to Proposition 3.3.5), these calculations determine their trace functions. We justify this next.

Since \mathcal{E} and \mathcal{E}' are $\mathrm{SL}(2)_k$ -equivariant, there is an isomorphism

$$\Phi: c^* \mathcal{L} \cong p^* \mathcal{L}$$

where \mathcal{L} is either of those local systems, c the conjugation by $\mathrm{SL}(2)_k$ morphism, and $p: G_k \times \mathcal{U}_{0,k} = \mathrm{SL}(2)_k \times \mathcal{U}_{0,k}$ the canonical projection. Let $g \in G_k(k)$ and define morphisms

$$c(g): \mathcal{U}_{0,k} \longrightarrow \mathcal{U}_{0,k}$$

and

$$p(g): \mathcal{U}_{0,k} \longrightarrow \mathcal{U}_{0,k}$$

by the following commutative diagram.

$$\begin{array}{ccc}
 & G_k \times \mathcal{U}_{0,k} & \xrightarrow{c \text{ (resp. } p)} \mathcal{U}_{0,k} \\
 & \uparrow g \times \text{id}_{\mathcal{U}_{0,k}} & \nearrow c(g) \text{ (resp. } p(g)) \\
 h \curvearrowleft & \text{Spec}(k) \times \mathcal{U}_{0,k} & \\
 & \uparrow \wr & \\
 & \mathcal{U}_{0,k} &
 \end{array}$$

The object in the middle of the left side is simply the closed subscheme $\{g\} \times \mathcal{U}_{0,k}$; the map $c(g)$ is just the conjugation-by- g map; and $p(g)$ is the composition of the inclusion of $\mathcal{U}_{0,k}$ into the product with the canonical projection, so in fact, for all g , $p(g) = \text{id}_{\mathcal{U}_{0,k}}$. The equivariance morphism Φ restricts to these morphisms, since

$$\begin{aligned}
 c(g)^* \mathcal{L} &= (c \circ h)^* \mathcal{L} \\
 &= h^*(c^* \mathcal{L}) \\
 &\cong h^*(p^* \mathcal{L}) && (\text{by } h^* \Phi) \\
 &= (p \circ h)^* \mathcal{L} \\
 &= p(g)^* \mathcal{L}.
 \end{aligned}$$

Restrict Φ to the closed subscheme $\{g\} \times \mathcal{U}_{0,k}$. Composing the inclusion of that closed subscheme with the isomorphism $\{g\} \times \mathcal{U}_{0,k} \cong \mathcal{U}_{0,k}$ determines an

isomorphism

$$\alpha(g)^*\mathcal{L} \cong p(g)^*\mathcal{L},$$

Because $p(g) = \text{id}_{\mathcal{U}_{0,k}}$, this isomorphism can be rewritten as

$$c(g)^*\mathcal{L} \cong \mathcal{L}.$$

Arising as they do from a single isomorphism of schemes, the family of these isomorphisms as g varies over $G_k(k)$ are all compatible. The trace of Frobenius function depends only on the isomorphism class of the complex, so

$$t_{\text{Fr}}^{\mathcal{L}} = t_{\text{Fr}}^{c(g)^*\mathcal{L}}.$$

In addition (as in Laumon [18, 1.1.1.4]),

$$t_{\text{Fr}}^{c(g)^*\mathcal{L}} = t_{\text{Fr}}^{\mathcal{L}} \circ c(g).$$

Thus, the calculations above entirely determine the trace function, and

$$t_{\text{Fr}}^{\mathcal{L}}(x) = \begin{cases} 1 & x \text{ is in the orbit of } a, \\ -1 & x \text{ is in the orbit of } a', \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we can move to establishing the convergence of $\Theta_{\mathcal{L}}$. For this, we refer to Table 2 in Chapter 15 of Digne and Michel [11]. The fifth and sixth rows of this table correspond to characters of irreducible cuspidal representations of

$G(k)$ that we will denote χ_+ and χ_- . Further inspection of the table reveals that

$$t_{\text{Fr}}^{\mathcal{L}} = c(\chi_+ - \chi_-),$$

where c is a constant we can calculate but will safely ignore.

Let σ_+ and σ_- be the irreducible cuspidal representations corresponding to χ_+ and χ_- , and let π_+ and π_- be depth-zero supercuspidal representations of $G(K)$ obtained from σ_+ and σ_- via compact induction, inflated from the standard maximal compact subgroup (denoted $X(\mathcal{O}_K)$ in Chapter 4) of $G(K)$, while π'_+ and π'_- denote the supercuspidals inflated from the nonstandard maximal compact subgroup. These four depth-zero supercuspidal representations are in fact the residents of the lone quaternary L-packet for $G(K)$. Harish-Chandra [14, (§5)] has described how to associate characters (in fact, distributions on $G(K)$) to such representations, which we denote $\Theta_{\pi_{\pm}}$ and $\Theta_{\pi'_{\pm}}$. These characters are given by the following so-called Frobenius formula,

$$\begin{aligned}\Theta_{\pi_{\pm}}(f) &= \int_{G(K)} \int_{G(K)} f(ghg^{-1})f_{\pm}(h)dh, dg \\ \Theta_{\pi'_{\pm}}(f) &= \int_{G(K)} \int_{G(K)} f(ghg^{-1})f'_{\pm}(h)dh, dg\end{aligned}$$

(see, for example, Cunningham and Gordon [8]). The functions f_{\pm} and f'_{\pm} appearing in the formulæ are the locally constant, compactly supported functions on $G(K)$ obtained by inflating the characters χ_{\pm} to, respectively, the standard and nonstandard maximal compact subgroup, and then extending by zero (the same process that induced the function $f_{\mathcal{L}}$, recall). These integrals are known

to converge (by Harish-Chandra [14, §5]), so that the differences

$$\begin{aligned}\Theta_{\pi_+}(f) - \Theta_{\pi_-}(f) &= \int_{G(K)} \int_{G(K)} f(ghg^{-1}) f_{\circ_{\mathcal{E}}}(h) dh, dg \\ \Theta_{\pi'_+}(f) - \Theta_{\pi'_-}(f) &= \int_{G(K)} \int_{G(K)} f(ghg^{-1}) f_{\circ_{\mathcal{E}'}}(h) dh, dg\end{aligned}$$

likewise converge. These are the integrals $\Theta_{\mathcal{E}}$ and $\Theta_{\mathcal{E}'}$, respectively, and so they converge.

2. This is immediate from the definition, which includes linear combinations of distributions obtained as characters of admissible representations, in view of the fact that $\Theta_{\mathcal{L}}$ is equal to one of the differences of integrals above.

□

5.3 Distributions on the Lie Algebra

Further significant properties appear after transporting $\Theta_{\mathcal{E}}$ and $\Theta_{\mathcal{E}'}$ to the Lie algebra of $G(K)$,

$$\mathfrak{g} := \mathfrak{sl}(2)_K = \text{Spec}(K[a, b, c, d]/(a + d)).$$

To do so, we will make use of the modified Cayley transform, which is defined on points by

$$\begin{aligned}\text{cay}: \mathfrak{g}(K) &\longrightarrow G(K) \\ g &\longmapsto \left(1 + \frac{g}{2}\right) \left(1 - \frac{g}{2}\right)^{-1}.\end{aligned}$$

The transform gives an isomorphism between topologically nilpotent elements of $\mathfrak{g}(K)$ and topologically unipotent elements in $G(K)$ (see Cunningham & Gordon [8]). Precomposition by cay gives an isomorphism of Hecke algebras that we also call cay :

$$\text{cay}: \mathcal{C}(\mathfrak{g}(K)) \longrightarrow \mathcal{C}(G(K)).$$

Definition 5.3.1. For $\mathcal{L} \in \{\mathcal{E}, \mathcal{E}'\}$ and any $f \in \mathcal{C}(\mathfrak{g}(K))$, let

$$D_{\mathcal{L}}(f) := \Theta_{\mathcal{L}}(\text{cay}(f)).$$

By Proposition 5.2.1, this defines distributions on $\mathfrak{g}(K)$.

Recall that the Fourier transform on $\mathfrak{g}(K)$ is defined by first making a choice of character $\psi: K \rightarrow \overline{\mathbb{Q}}_{\ell}^{\times}$ and setting, for any $X, Y \in \mathfrak{g}(K)$,

$$\psi(X, Y) := \psi(\text{Trace}(XY)).$$

Then the Fourier transform of any $f: \mathfrak{g}(K) \rightarrow \overline{\mathbb{Q}}_{\ell}^{\times}$ in $\mathcal{C}(\mathfrak{g}(K))$ is

$$\hat{f}(X) := \int_{\mathfrak{g}(K)} f(Y) \psi(X, Y) dY.$$

The Fourier transform of a distribution D on $\mathfrak{g}(K)$ is then

$$\hat{D}(f) := D(\hat{f}), \quad f \in \mathcal{C}(\mathfrak{g}(K)).$$

Proposition 5.3.2. *Let $\hat{D}_{\mathcal{L}}$ be the Fourier transform of $D_{\mathcal{L}}$ for $\mathcal{L} \in \{\mathcal{E}, \mathcal{E}'\}$. There exists a $\lambda \in \overline{\mathbb{Q}}_{\ell}$ such that $\hat{D}_{\mathcal{E}} = \lambda D_{\mathcal{E}}$ and $\hat{D}_{\mathcal{E}'} = \lambda D_{\mathcal{E}'}$. Moreover, $D_{\mathcal{E}}$ and $D_{\mathcal{E}'}$*

are linearly independent in the space of all admissible distributions on $\mathcal{C}(\mathfrak{g}(K))$.

Proof. 1. For any $f \in \mathcal{C}(\mathfrak{g}(K))$,

$$\begin{aligned}\hat{D}_{\mathcal{L}}(f) &= D_{\mathcal{L}}(\hat{f}) \\ &= \int_{G(K)} \int_{\mathfrak{g}(K)} \hat{f}(y^{-1}Yy) \text{cay}(f_{\mathcal{L}})(Y) dY dy \\ &= \int_{G(K)} \int_{\mathfrak{g}(K)} f(y^{-1}Yy) \widehat{\text{cay}(f_{\mathcal{L}})}(Y) dY dy.\end{aligned}$$

In Waldspurger [34, II.4,p. 34], the character $\chi_+ - \chi_-$ appears, denoted by ${}^\circ f$, and is identified as a cuspidal function; by the first corollary in Lusztig [26, § 10], such functions are eigenfunctions of the Fourier transform on $\mathfrak{g}(K)$. Thus, by Proposition 1.13 in Cunningham and Hales [9],

$$\hat{f}_{\mathcal{L}} = \lambda f_{\mathcal{L}}$$

(The proposition gives the value of λ explicitly). Therefore,

$$\hat{D}_{\mathcal{L}} = \lambda D_{\mathcal{L}}.$$

2. Follows from Table 12 in Cunningham and Gordon [8].

□

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